Lecture 4

Sept. 2nd, 2011

Today's Outline:

I Last Lecture:
Finished damped oscillator and ...

II Driven damped oscillator

III Resonance

I. We found the EOM:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t) \equiv \frac{F(t)}{m}$$

Decided to focus on the driving force (per unit mass):

$$f(t) = f_0 \cos(\omega t)$$

ω_0 = natural frequency

ω = driving frequency

This is an inhomogeneous linear diff. eq. S'pose $x_h(t)$ is the general solution to the associated homogeneous eq. then

$$D x_h = 0$$

where

$$D \equiv \frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2.$$

S'pose further that $x_p(t)$ is any old ("particular") solution to the inhomog. eq.

$$D x_p(t) = f(t)$$

Then $x_h + x_p$ solves the inhomog. eq

$$D(x_h + x_p) = D x_h + D x_p$$
$$= 0 + f(t)$$

and is, in fact, the general solution. [In HW you will show it doesn't matter what x_p you use.]

So, to solve driven case we focus on finding a particular solution. <u>Do this by hook or by crook</u>.

The standard guess has shown how nice complex exp. are. So, view this EOM as the real part of

$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\omega t}$$

with $z(t) \equiv x(t) + iy(t)$.

The imaginary part is

$$\ddot{y} + 2\beta\dot{y} + \omega_0^2 y = f_0 \sin\omega t.$$

Try to guess a particular solution

$$z_p = C e^{i\omega t}$$

Physically this says that if we drive an oscillator at frequency ω we might expect it responds at the same freq.

Then,

$$\dot{z}_p = i\omega C e^{i\omega t} \text{ and } \ddot{z}_p = -C\omega^2 e^{i\omega t}$$

So,

$$-C\omega^2 \cancel{e^{i\omega t}} + 2\beta i\omega C \cancel{e^{i\omega t}} + \omega_0^2 C \cancel{e^{i\omega t}} = f_0 \cancel{e^{i\omega t}}$$

$$\Rightarrow C = \frac{f_0}{\omega_0^2 - \omega^2 + 2\beta i\omega}$$

That gives a particular solution(!), simplify it:

Like to write

$$z_p = C e^{i\omega t} = \underset{\text{real}}{A} e^{-i\delta} e^{i\omega t}$$

$$|e^{i\theta}|^2 = e^{i\theta} \cdot e^{-i\theta} = 1$$

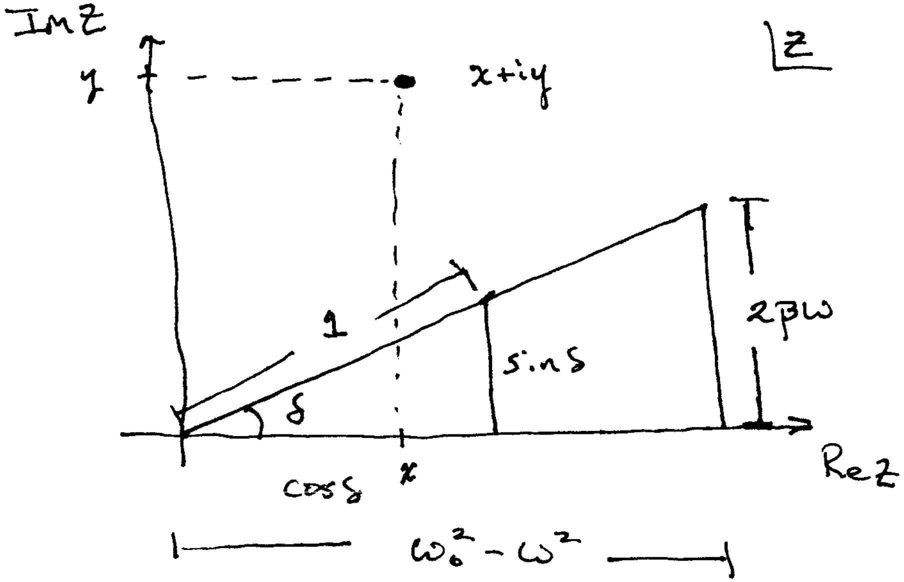
So,

$$|A|^2 = |C|^2 = CC^*$$

$$= \frac{f_0}{(\omega_0^2 - \omega^2) + 2\beta i\omega} \cdot \frac{f_0}{(\omega_0^2 - \omega^2) - 2\beta i\omega}$$

$$= \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

$$\Rightarrow \quad A = \frac{f_0}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}}$$



After transients have died out the motion is independent of initial conditions and the $A\cos(\omega t-\delta)$ term is called an <u>attractor</u>.

<u>III Resonance:</u>

Analyze the amplitude

$$A^2 = \frac{f_0^2}{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}$$

$$C = A e^{-i\delta}$$

$$\Rightarrow \quad e^{i\delta} = \frac{A}{f_0}\left(\omega_0^2-\omega^2+2\beta i\omega\right)$$

real # rescales triangle

From diagram

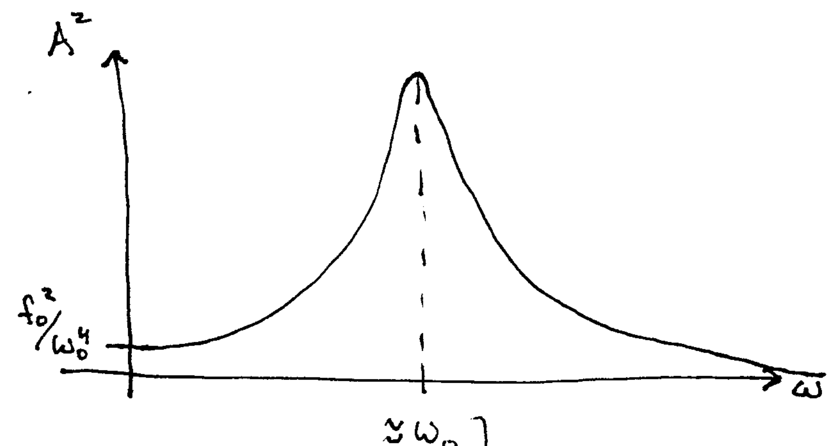
$$\tan\delta = \frac{2\beta\omega}{\omega_0^2-\omega^2}$$

General solution

not arbitrary

$$x(t) = A\cos(\omega t-\delta) + \underbrace{C_1 e^{r_1 t} + C_2 e^{r_2 t}}$$

Exponentially damped <u>transients</u>.



$$A^2_{max} \simeq \frac{f_0^2}{4\beta^2\omega_0^2}$$

show this:

A^2 maximized when denominator is a minimum, so it is near $\omega=\omega_0$, but

Let's calculate it:

$$\frac{d}{d\omega}\left((\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2 \right) = 0$$

$$\Rightarrow \quad \cancel{2}(\omega_0^2-\omega^2)(-\cancel{2}\cancel{\omega}) + \overset{4\,2}{\cancel{8}}\beta^2\cancel{\omega} = 0$$

$$\Rightarrow \quad \omega^2 - \omega_0^2 + 2\beta^2 = 0$$

$$\Rightarrow \quad \omega_* = \sqrt{\omega_0^2 - 2\beta^2} \overset{\text{call it}}{=} \omega_2$$

Indeed for small β

$$\omega_2 \approx \omega_0$$

Again assume ($\beta \ll \omega_0$) β is small.

Then for $\omega = \omega_0 \pm \beta$,

$$(\omega_0^2-\omega^2) = (\omega_0+\omega)(\omega_0-\omega)$$
$$\approx 2\omega_0(\mp\beta)$$

and

$$4\beta^2\omega^2 \approx 4\beta^2\omega_0^2 + O(\beta^3)$$

so

$$A^2(\omega_0 \pm \beta) \approx \frac{f_0^2}{(\mp 2\beta\omega_0)^2 + 4\beta^2\omega_0^2}$$

$$= \frac{f_0^2}{8\beta^2\omega_0^2} \approx \frac{1}{2} A_{max}^2$$

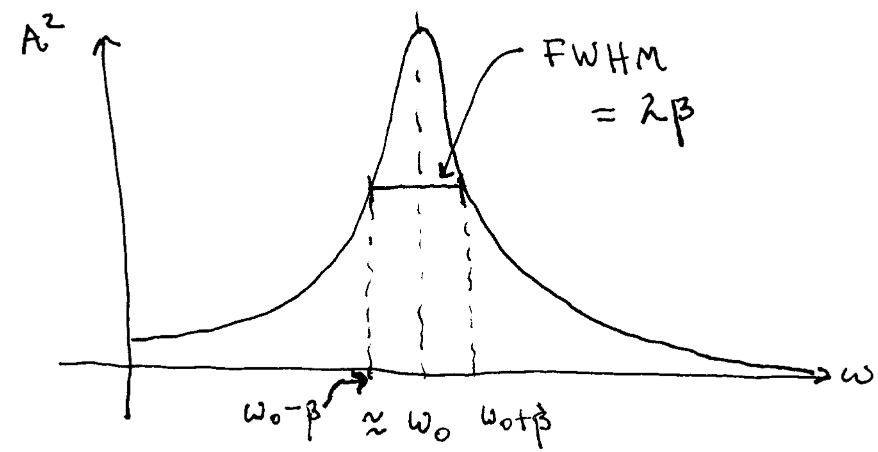
Then,

$$A_{max}^2 \approx A^2(\omega_0)$$
$$= \frac{f_0^2}{4\beta^2\omega_0^2}.$$

We've characterized the height of the resonance. We'd also like to characterize its width. This is done with

FWHM = "Full Width at Half Max"

or HWHM = "Half Width at Half Max".



A dimensionless measure of the sharpness of the resonance is the <u>Quality factor</u>

$$Q = \frac{\omega_0}{2\beta}$$

Large value of Q $\Rightarrow$ narrow resonance.

Examples of resonance:

Electrical: many many, but e.g. a radio tuner.

Optical Spectra:
e.g. The absorption spectrum of the sun

Nuclear: NMR = Nuclear magnetic resonance good for medical imaging.