day's Outline:

I. Motivation

II. Euler's Method in the calculus of variations

Lecture 5
Sept. 7th, 2011

I. New approach to Mechanics called a variational principle. Based on new mathematics: the calculus of variations.

Because this approach is new and its tools are new it can be easy to lose the forest for the trees. This motivation will try to mitigate this.

Recall that Newton's laws are a set of axioms:

I. velocity is const. unless body is acted on by a force

II The acceleration is given by
$$\vec{a} = \vec{F}/m$$

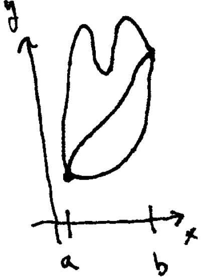
III Mutual forces of action and reaction are equal, opposite and collinear.

They have been promoted to the status of laws because of their incredible predictive successes.

But <u>other</u> axioms are possible.

A variational principle expresses the idea that the correct motion of a system can be predicted be extremizing (maximizing, minimizing, saddling) an integral
$$I = \int^{a} f(x, y(x), y'(x))\, dx.$$

Eventually the integral itself will take on physical meaning (particularly in Quantum Mechanics) but for now we will focus on the path ~~y~~ $y(x)$ that we feed to the integral. By changing this path we change the value of I and our immediate goal will be to find ~~those paths~~ a condition ~~that extremize~~ on the path that ~~ex~~ gaurentees it will extremize I.



Diff. paths give diff. integrals.

Physically our independent variable will be t and our path will be, e.g. $x(t)$. We will find an integral $S[x(t)]$ which is extremized when $x(t)$ satisfies the physical equations of motion. Thus the E.O.M. will become the condition that gaurentees S is extremized.

Strategy: Math first then physics.

Reason: Allows us to seperate the
(for strategy)
Calculation and its interpretation.

Notation: I for integral
x for independent variable
y for depend. variable
$y(x)$ for the path.

$\left.\right\}$ Appropriate ~~for~~ math

~~Also~~ We will adopt physical notation starting next week.

Why bother with all of this?

Many reasons but one important one is that it will free us from the shackles of Cartesian coord. systems. Adapt our coord.s to the symmetry of the system. I'll list more as we go.

Today: Euler's method (not in your book)

Friday: Lagrange's method. (in your book)

II Euler's Method

Suppose that the curve in Figure 1. extremizes the integral

$$I = \int_0^a f(x, y(x), y'(x))\,dx$$

We want to find an equation determining this curve $y(x)$.

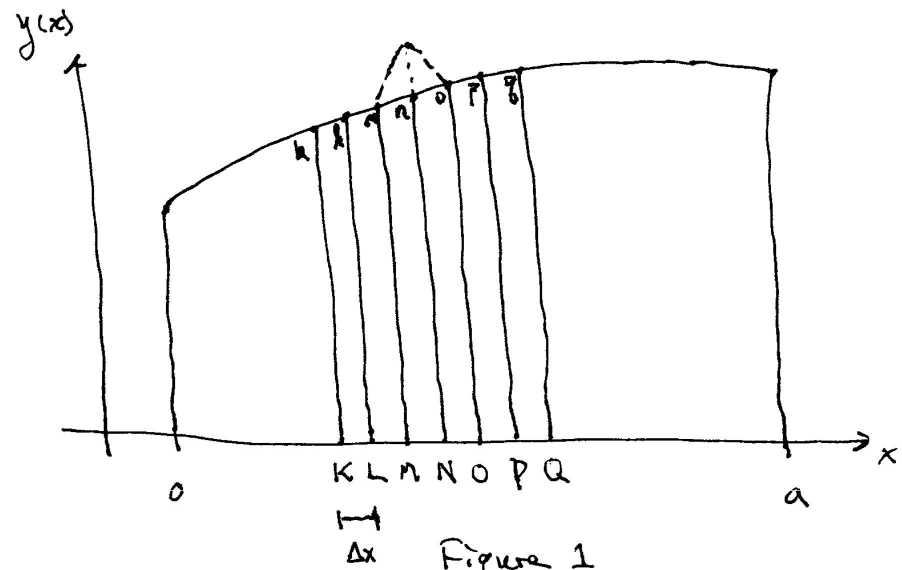


Figure 1

We will proceed in an approximate manner, similar to a Riemann sum in calculus:

(1) Divide the interval between $x=0$ and $x=a$ into many subintervals of width Δx

(2) Approximate the integral by a sum

$$\int_0^a f(x,y,y')\,dx \approx \sum_0^a f(x,y,y')\,\Delta x$$

In each term of this sum evaluate f at the initial point of the subinterval x_N, $y(x_N) \equiv y_n$

(3) Approximate the derivative ~~slope~~ $y' = dy/dx$ by the slope of the straight line connecting the initial and final points of the subinterval. That is,

$$y'_m \equiv \frac{y_n - y_m}{\Delta x} \approx y'(x_M).$$

All of these approximations become excellent in the limit of many subintervals, $\Delta x \to 0$.

Now, let's change y_n (see dashed lines in Fig. 1). This changes the sum but only the terms involving y_n, which turn out to be the m term and the n term. The sum no longer depends on the whole path just the values of f at the discrete points x_N, y_n and y_n', so we can extremize it with regular calculus; the condition is

$$\frac{\partial}{\partial y_n}\left(\sum_0^a f(x,y,y')\,\Delta x\right) = 0.$$

~~Before we calculate this it is useful to review the chain rule.~~

If you're rusty on the chain rule, see the notes on our website. We'll use it here to do this calculation. Ok, let's do it

$$\frac{\partial}{\partial y_n}\left(\sum_0^a f(x,y,y')\,\Delta x\right) =$$

terms not involving y_n

$$\frac{\partial}{\partial y_n}\Big(\cdots + f\left(x_M, y_m, \frac{y_n - y_m}{\Delta x}\right)\Delta x + f\left(x_N, y_n, \frac{y_o - y_n}{\Delta x}\right)\Delta x + \cdots\Big)$$

chain rule

$$= \frac{\partial f}{\partial y_m'}\frac{\partial}{\partial y_n}\left(\frac{y_n - y_m}{\Delta x}\right)\cdot\Delta x + \frac{\partial f}{\partial y_n}\cdot\Delta x + \frac{\partial f}{\partial y_n'}\frac{\partial}{\partial y_n}\left(\frac{y_o - y_n}{\Delta x}\right)\cdot\Delta x$$

$$= \frac{\partial f}{\partial y_m'}\cdot\frac{1}{\Delta x}\cdot\Delta x + \frac{\partial f}{\partial y_n}\cdot\Delta x + \frac{\partial f}{\partial y_n'}\left(-\frac{1}{\Delta x}\right)\cdot\Delta x$$

Setting this equal to zero and rearranging gives

$$\frac{\partial f}{\partial y_n}\cdot\Delta x - \left(\frac{\partial f}{\partial y_n'} - \frac{\partial f}{\partial y_m'}\right) = 0$$

Now, dividing by Δx gives

$$\frac{\partial f}{\partial y_n} - \frac{1}{\Delta x}\left(\frac{\partial f}{\partial y_n'} - \frac{\partial f}{\partial y_m'}\right) = 0$$

In the limit as $\Delta x \to 0$ this becomes

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0}$$

the Euler-Lagrange Equations

Note the distinctions between partial and total derivatives here.

Euler's method is nice because it's so closely tied to the geometry. Next lecture we will look at Lagrange's method which is mathematically slicker.