Today's Outline Lecture 6 Sept. 9th, 2011

I Last Step of Euler's Method

II Example of Euler's method

III Lagrange's method in the calculus of variations

IV Examples: Geodesics on the cylinder

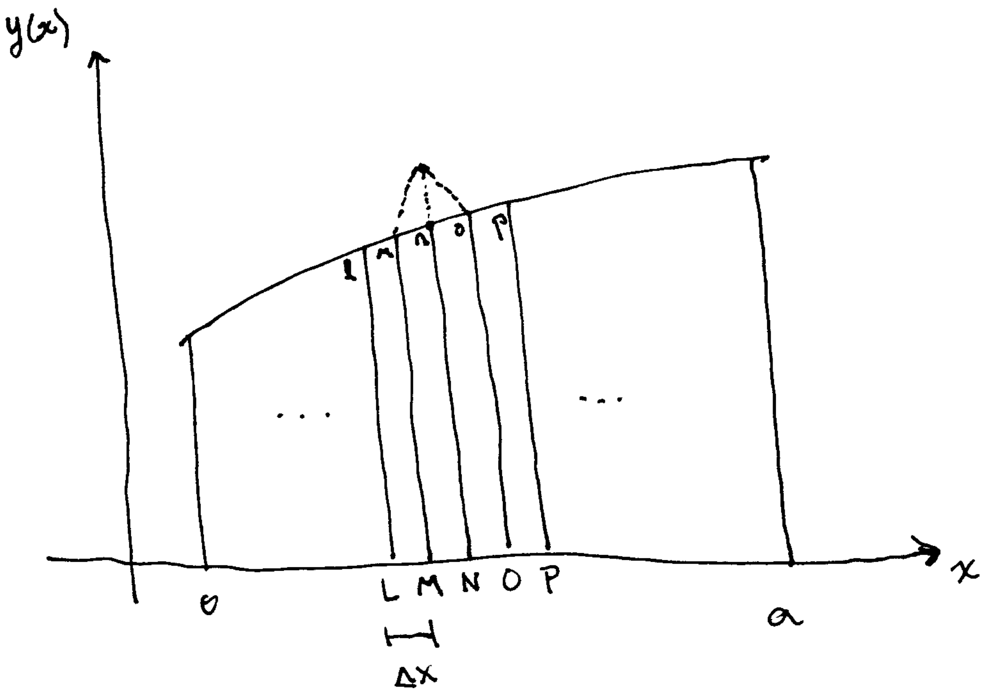


Figure 1.

I.

Last time we arrived at the equation

$$\frac{\partial f}{\partial y_n} - \frac{1}{\Delta x}\left(\frac{\partial f}{\partial y_n'} - \frac{\partial f}{\partial y_m'}\right) = 0$$

as a condition for the sum

$$\sum_0^a f(x, y, y')\,\Delta x$$

to be extremized with respect to variations of y_n. See Figure 1 for notation.

In the limit as $\Delta x \to 0$ this condition becomes

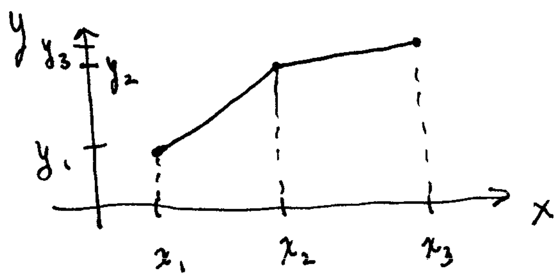
$$\frac{\partial f}{\partial y(x)} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'(x)}\right) = 0 .$$

Notice that our derivation holds for any point of the curve, that is an arbitrary x and $y(x)$, hence all of them. Thus in using Euler's method you can always focus on a small segment of the curve.

II Let's do an example to illustrate this.

What is the shortest path connecting two points in the xy-plane?

Focus first on two nearby point (x_1, y_1) and (x_3, y_3)



So we impose ~~calculate~~,

$$\frac{\partial L}{\partial y_2} = 0$$

$$\Rightarrow \frac{1}{2}\frac{1}{\sqrt{\Delta x^2 + (y_2-y_1)^2}} \cdot 2(y_2-y_1) + \frac{1}{2}\frac{1}{\sqrt{\Delta x^2+(y_3-y_2)^2}}$$

$$\times\ 2(y_3-y_2)\cdot(-1) = 0$$

$$\Rightarrow \frac{y_2-y_1}{\sqrt{\Delta x^2 + (y_2-y_1)^2}} = \frac{y_3-y_2}{\sqrt{\Delta x^2+(y_3-y_2)^2}}$$

The length of the path connecting them depends on how we choose the intermediate point (x_2, y_2).

This length is just

$$L = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} + \sqrt{(x_3-x_2)^2+(y_3-y_2)^2}$$

$$= \sqrt{\Delta x^2 + (y_2-y_1)^2} + \sqrt{\Delta x^2 + (y_3-y_2)^2}$$

We want to choose y_2 such that L is a minimum.

Factor out Δx^2, ~~and~~

$$\frac{(y_2-y_1)/\Delta x}{\sqrt{1 - \frac{(y_2-y_1)^2}{\Delta x^2}}} = \frac{(y_3-y_2)/\Delta x}{\sqrt{1 - \frac{(y_3-y_2)^2}{\Delta x^2}}}$$

and we notice these expressions are equal if

$$\left(\frac{y_2-y_1}{\Delta x}\right) = \left(\frac{y_3-y_2}{\Delta x}\right),$$

that is,

the slope is constant!

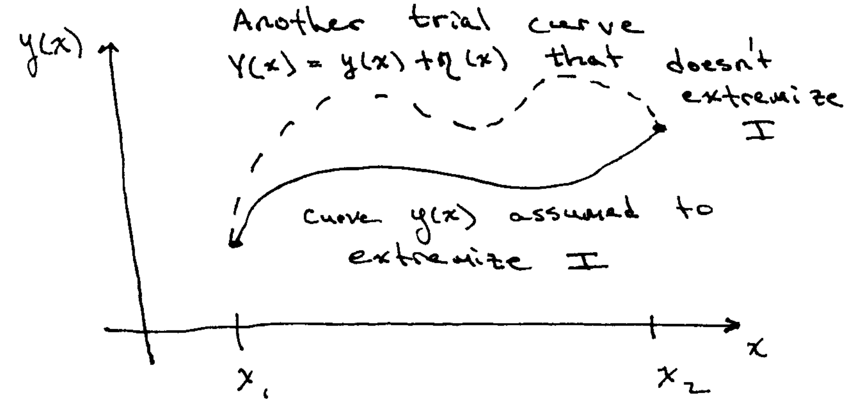
It's a straight line.

Imposing this condition, constant slope, on each segment of the curve gives a straight line for the whole curve.

III Lagrange's Method in the Calculus of variations

Want to find the curve $y(x)$ that extremizes

$$I = \int_{x_1}^{x_2} f(x, y(x), y'(x))\,dx$$



Lagrange also wants to turn this into ~~anoth~~ a calculus problem. He found a very clever way to do this:

Write your trial solution

$$Y(x) = y(x) + \alpha\,\eta(x)$$

in terms of a parameter α. Now you can turn on your variation of the whole curve just by changing α. In particular I becomes a normal function of α, $I(\alpha)$. So the condition that I is extremized is just

$$\frac{dI}{d\alpha} = 0$$

[Subtlety: We assume $y(x)$ is the "right" curve so that we know where to evaluate the derivative, namely,

$$\left.\frac{dI}{d\alpha}\right|_{\alpha=0} = 0 .$$]

Lets do the calculation.

If $Y(x) = y(x) + \alpha\,\eta(x)$ then

$$Y'(x) = y'(x) + \alpha\,\eta'(x).$$

So,

$$I(\alpha) = \int_{x_1}^{x_2} f(x, Y, Y')\,dx$$

$$= \int_{x_1}^{x_2} f(x, y(x)+\alpha\eta, y'+\alpha\eta')\,dx$$

Then,

$$\frac{dI}{d\alpha} = \frac{d}{d\alpha}\int_{x_1}^{x_2} f\,dx$$

$$= \int_{x_1}^{x_2} \frac{\partial f}{\partial \alpha}\,dx$$

taking the derivative inside the integral makes it a partial.

To simplify the second term we use integration by parts

$$\int_{x_1}^{x_2} \frac{d}{dx}\big(f(x)\,g(x)\big)\,dx = f(x)g(x)\Big|_{x_1}^{x_2}$$

"The integral is the antiderivative."

But also

$$\int_{x_1}^{x_2} \frac{d}{dx}(f\,g)\,dx = \int_{x_1}^{x_2} f'g\,dx + \int_{x_1}^{x_2} f\,g'\,dx$$

Putting these together

$$\int_{x_1}^{x_2} \frac{df}{dx}\,g\,dx = f(x)g(x)\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} f\cdot\frac{dg}{dx}\,dx$$

$$\frac{\partial}{\partial\alpha}\left[f(x, y+\alpha\eta, y'+\alpha\eta')\right]$$

$$\overset{\text{chain rule}}{=} \frac{\partial f}{\partial y}\cdot\frac{\partial}{\partial\alpha}(y+\alpha\eta) + \frac{\partial f}{\partial y'}\,\frac{\partial}{\partial\alpha}(y'+\alpha\eta')$$

$$= \frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta'$$

So,

$$\frac{dI}{d\alpha} = \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta'\right)dx$$

"You can switch the derivative at the cost of a minus sign and a boundary term."

Then

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\,\frac{d\eta}{dx}\,dx = \underbrace{\left(\frac{\partial f}{\partial y'}\eta\right)\Big|_{x_1}^{x_2}}_{=0} - \int_{x_1}^{x_2}\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta\,dx$$

and because of assumed bndry conditions.

$$\frac{dI}{d\alpha} = \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right)\eta\,dx$$

Want to set

$$\frac{dI}{d\alpha} = 0.$$

The pinnacle of the α cleverness is that this equation has to hold for <u>all</u> η. So what can we say when

$$\int_{x_1}^{x_2} g(x)\eta(x) = 0$$

for all η? Well suppose $g \neq 0$ and choose η to be positive whenever g is and similarly for negative values. Then $g(x)\eta(x) > 0$ always and

$$\int_{x_1}^{x_2} g(x)\eta(x)\,dx \neq 0$$

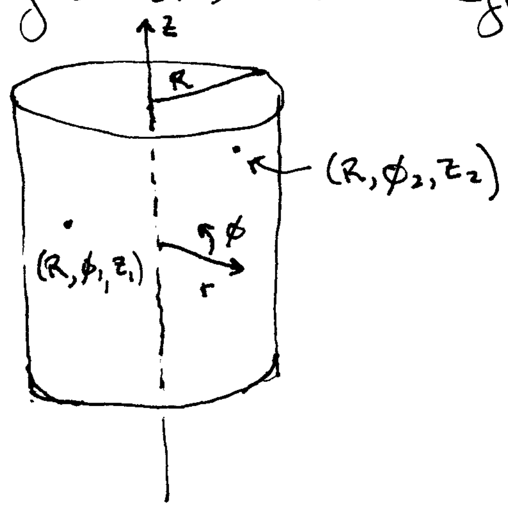
a contradiction! $\Rightarrow$ $\boxed{g(x) = 0}$

<u>So</u>,

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0. \quad \text{E-L Eq.s again!}}$$

That completes our second proof. Can go to examples (at last!).

Find the geodesics on a cylinder.



Describe as $\phi(z)$.

$$ds^2 = R^2 d\phi^2 + dz^2$$

$$\Rightarrow \quad ds = \sqrt{R^2\left(\frac{d\phi}{dz}\right)^2 + 1}\; dz$$

$$L = \int_{z_1}^{z_2} \underbrace{\sqrt{R^2(\phi')^2 + 1}}_{f(z,\phi,\phi') = f(\phi')}\; dz$$

E-L eq.

$$\cancelto{0}{\frac{\partial f}{\partial \phi}} - \frac{d}{dz}\left(\frac{\partial f}{\partial \phi'}\right) = -\frac{d}{dz}\left(\frac{\partial f}{\partial \phi'}\right) = 0$$

$$\Rightarrow \quad \frac{\partial f}{\partial \phi'} = \text{const.} \overset{\text{call it}}{\equiv} C_1$$

$$\Rightarrow \frac{1}{2} \frac{1}{\sqrt{R^2 \phi'^2 + 1}} \cdot 2R^2 \phi' = C_1$$

$$\Rightarrow \quad R^4 \phi'^2 = C_1^2 (R^2 \phi'^2 + 1)$$

$$\Rightarrow (R^4 - C_1^2 R^2)\, \phi'^2 = C_1^2$$

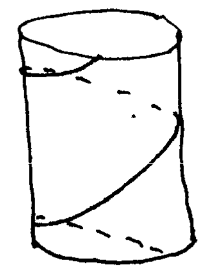
$$\Rightarrow \quad \phi' = \text{const.} \overset{\text{call it}}{\equiv} C$$

Then

$$\phi(z) = Cz + \overset{\text{constant}}{K}$$

$$\left.\begin{aligned} \phi(z_1) &= Cz_1 + K = \phi_1 \\ \phi(z_2) &= Cz_2 + K = \phi_2 \end{aligned}\right\} \text{solve } C \text{ and } K$$

This is the equation of a helix:

.