Lecture 9
Sept. 16th, 2011

Today's Outline:

I. Loose Ends

II. Constraints

I. In lecture 7 (Monday) we saw that the E.-L. eq.s took the same form in every coordinate system. Last lecture we also saw that they were very efficient. One can't help but ask: why? (!) This is a broad question but one answer is that $\mathcal{L}$ is a Scalar. This is what allowed us to change coordinates and still end up with

$$S = \int \mathcal{L}\, dt$$

and the E.-L. equations. In modern physics we exploit the Scalar nature of $\mathcal{L}$ to the hilt; we write down every known scalar consistent with the symmetries of the system and throw them into $\mathcal{L}$. More on this connection next week.

II. The second great advantage of the Lagrangian formulation is its seemless encorporation of constraints. Let's show this.

Setup: Definition of degrees of freedom (D.O.F.) in general:

# of D.O.F. = # of coord.s that can be independently varied in a small displacement

e.g. pendulum 1 D.O.F.

N particles in 3D   3N D.O.F

When the # of D.O.F. of N particles in 3D is less than 3N, we say the system is constrained.

Often

D.O.F = # generalized coord.s used to describe system

in this case we say the constraints are <u>holonomic</u> (Beware other defn.s exist.)

We will focus on holonomic constraints but if you get curious ask me about non-holonomic systems (there was a big breakthrough just this year).

Goal: Prove that E.-L. equations hold even for systems with constraints:

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) \quad i=1,\dots,n$$

n = # gen. coord.s = # D.O.F.

Proof has two steps:

<u>Step 1</u>: The "What's so special about (holonomic) constraint forces?" step.

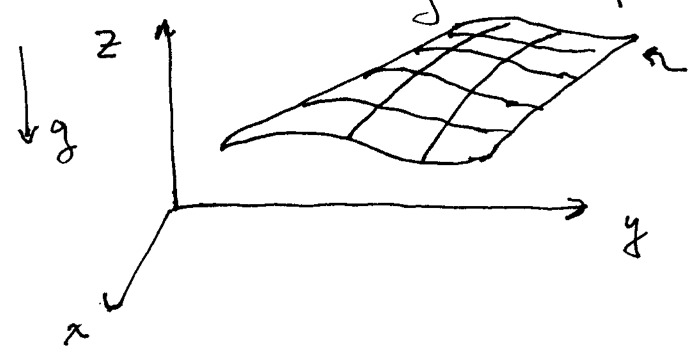
<u>Answer</u>: They do no work.

<u>Step 2</u>: The "oh, duh, it's exactly the same as before" step.

More intuitively holonomic constraints are geometric conditions we know the system will obey but for which we don't know the force equations.

E.g. The bead on a circular hoop we did last lecture.

Or a particle confined to move on a curved surface. We'll take this as our motivating example



coord.s on surface q_1 and q_2.

# of D.O.F. = 2

~~Ex~~ Step 1: In constrained problems there are two kinds of forces: applied forces (e.g. gravity) and constraint forces (e.g. the normal force due to the surface). Your book calls the applied forces "nonconstraint" forces. Give 'em names

$\vec{F}$ = applied forces

$\vec{F}_{cstr}$ = constraint forces

$$\vec{F}_{tot} = \vec{F} + \vec{F}_{cstr}.$$

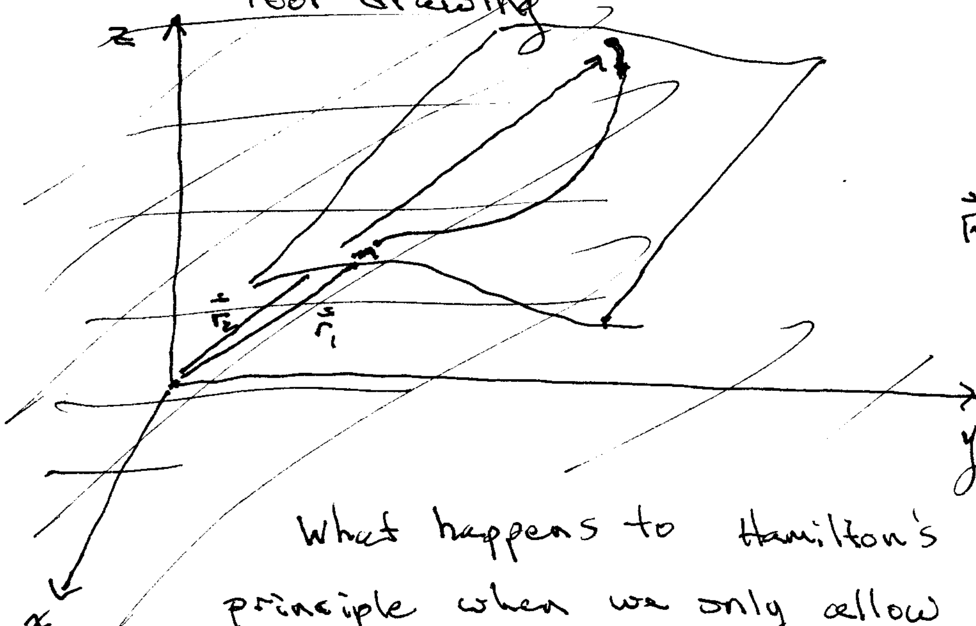
We will assume that all of the applied forces are derivable from a potential, so that,

$$\vec{F} = -\vec{\nabla}\, U(\vec{r}, t)$$

Let

$$\mathcal{L} = T - U$$

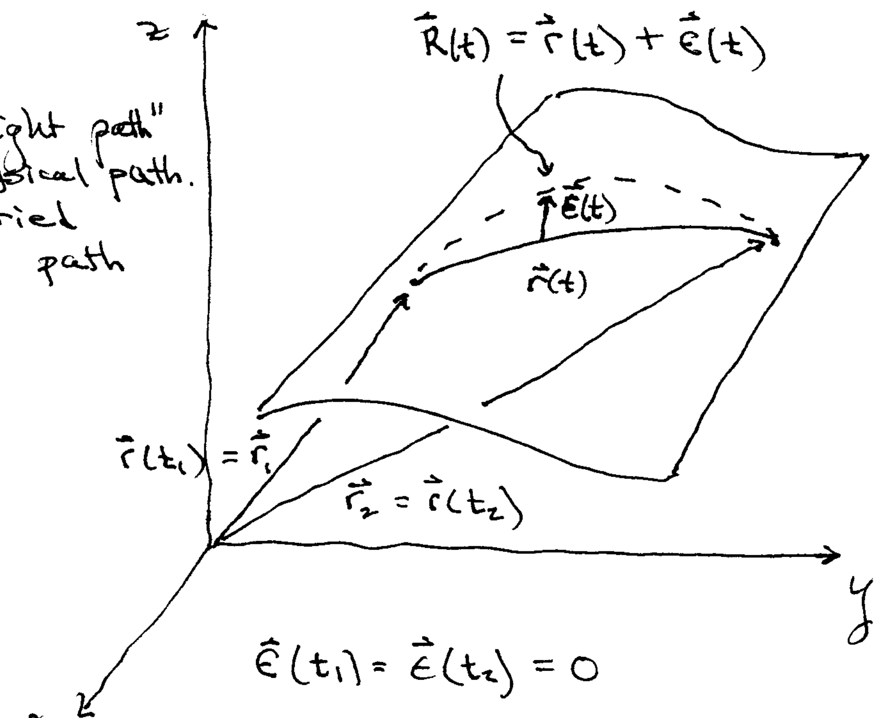
and notice that this excludes the constraint forces because of our definition of U.



What happens to Hamilton's principle when we only allow ourselves to vary over paths contained in the constraint surface?

$\vec{r}$ = "right path" = physical path.

$\vec{R}$ = varied path



$$\vec{\epsilon}(t_1) = \vec{\epsilon}(t_2) = 0$$

Let
$$S = \int_{t_1}^{t_2} \mathcal{L}(\vec{R}, \dot{\vec{R}}, t)\,dt,$$
$$S_0 = \int_{t_1}^{t_2} \mathcal{L}(\vec{r}, \dot{\vec{r}}, t)\,dt,$$
and
$$\delta S = S - S_0$$
← change in the integral S due to a change in path.

This boils down to
$$\delta \mathcal{L} = \mathcal{L}(\vec{R}, \dot{\vec{R}}, t) - \mathcal{L}(\vec{r}, \dot{\vec{r}}, t)$$
$$= \mathcal{L}(\vec{r}+\vec{\epsilon}, \dot{\vec{r}}+\dot{\vec{\epsilon}}, t) - \mathcal{L}(\vec{r}, \dot{\vec{r}}, t)$$

and $O(\epsilon^2)$ means "terms of order ϵ^2 or higher" here we're also using it for $\dot{\epsilon}^2$ or higher.

Then
$$\delta S = \int_{t_1}^{t_2} \delta\mathcal{L}\,dt$$
$$= \int_{t_1}^{t_2} \left[m\dot{\vec{r}}\cdot\dot{\vec{\epsilon}} - \vec{\epsilon}\cdot\vec{\nabla}U\right] dt$$
Integrate by parts ↓
$$= -\int_{t_1}^{t_2} \vec{\epsilon}\cdot\left[m\ddot{\vec{r}} + \vec{\nabla}U\right] dt$$

Now,
$$\mathcal{L} = \frac{1}{2} m \dot{\vec{r}}^2 - U(\vec{r}, t)$$
So
$$\delta\mathcal{L} = \frac{1}{2} m\left[(\dot{\vec{r}}+\dot{\vec{\epsilon}})^2 - \dot{\vec{r}}^2\right] - \left[U(\vec{r}+\vec{\epsilon}, t) - U(\vec{r}, t)\right]$$
$$= m\dot{\vec{r}}\cdot\dot{\vec{\epsilon}} - \vec{\epsilon}\cdot\vec{\nabla}U + O(\vec{\epsilon}^2)$$

The second term comes from multi-dimensional Taylor thm:
$$U(\vec{r}+\vec{\epsilon}, t) = U(\vec{r}, t) + \vec{\epsilon}\cdot\nabla U(\vec{r}, t) + \cdots$$

$$m\ddot{\vec{r}} = \vec{F}_{tot} = \vec{F} + \vec{F}_{cstr}$$
$$= -\vec{\nabla}U + \vec{F}_{cstr}$$
$$\Rightarrow \quad \delta S = -\int_{t_1}^{t_2} \left(\vec{\epsilon}\cdot\vec{F}_{cstr}\right) dt$$
but $\vec{\epsilon}\cdot\vec{F}_{cstr} = 0$ ($\vec{\epsilon}$ is tangent to constraint surface)
$$\Rightarrow \quad \delta S = 0$$

Hamilton's principle still holds!

Next time: Use $\delta S = 0$ to derive E.-L. eq.s w/ constraints