Physics 105 Lecture Notes: An Introduction to Symmetries and Conservation Laws

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Here we elaborate and expand on the material of Taylor, *Classical Mechanics*, section 7.8. We explore the connections between continuous symmetries of Lagrangian dynamical systems and dynamical invariants, formalized in the celebrated Noether's Theorem.

Also, congratulations to Hal and family, and best wishes to all! Hopefully the new Haggard will not make Hal too haggard.

IMPORTANT ANNOUNCEMENT: in-class midterm on Friday, September 23, 2011. Please bring a blank bluebook.

> *The mathmetical sciences particularly exhibit order, symmetry, limit, and proportion; and these are the greatest forms of the beautiful.*
>
> ARISTOTLE

> *Symmetry, as wide or as narrow as you may define its meaning, is one idea by which through the ages we have tried to comprehend and create order, beauty, and perfection.*
>
> HERMANN WEYL

I. MOTIVATION AND OVERVIEW

A. Why Do We Care About Symmetry?

The concept of *symmetry* is central to contemporary and classical physics. Symmetry is deeply tied to both our aesthetic judgements of the beauty of physical theories as well as out practical ability to solve problems. Symmetries may reveal important facts about nature, and simplify our descriptions of physical systems or their behavior. Themes of *symmetry*, *transformation*, and *conservation* will recur often in this class and throughout your studies of physics. Some go as far to say that every fundamental law of nature is an expression of symmetry in one form or another. Certainly the various forms of Noether's celebrated theorem, connecting symmetries of an action to conserved quantities under the evolution generated by it, as well as generalizations and thematically-related work by Sophus Lie, Élie Cartan, Albert Bäcklund, and others, provide foundation stones for building up both classical and quantum physics as now understood.

Although they might disagree on other issues, most working physicists would agree on this fundamental point. Steven Weinberg has written that "it is increasingly clear that the symmetry group of nature is the deepest thing we understand about the world today." Lee Smolin said that "The connection between symmetries and conservation laws is one of the great discoveries of twentieth century physics . But I think very few non-experts will have heard either of it or its maker — Emmy Noether, a great German mathematician. But it is as essential to twentieth century physics as famous ideas like the impossibility of exceeding the speed of light." Tsung-Dao lee said that "since the beginning of physics, symmetry considerations have provided us with an extremely powerful and useful tool in our effort to understand nature. Gradually they have become the backbone of our theoretical formulation of physical laws."

B. What Is Symmetry?

Derived from the ancient Greek for "agreement in proportion or arrangement," the word *symmetry* involves some notion that certain things remain unchanged even as others aspects do change. Imagine, as in class, that I hold up a perfect cue ball. You close your eyes momentarily, and when you open them, are unable to say whether I have rotated the cue ball or not. A symmetry is said to arise when something "looks the same" in some sense after transforming it in some way. Of course, we must flesh out what the "something" is, in what sense it "looks" the same, and under what sorts of changes or transformations. These will depend on the context at hand, and on what we hope the symmetry may accomplish for us.

In classical physics, the transformations are thought of as applying to a mechanical system — for example, physically shifting it, or rotating it, or imagining reflecting it in a mirror — or else to generalized coordinates or other observables and/or parameters describing the configuration and/or behavior of the system. The symmetry manifests as the *invariance* of certain properties of the system or its dynamical evolution, and/or of certain observable quantities associated with the system.

1. Active versus Passive Perspectives

Opportunities for confusion arise because at various stages we can often think of a transformation in an *active* sense, transforming the actual configuration or evolution of a physical system (for example, actually rotating the positions and velocities of a system of particles) or in a *passive* sense, transforming our description of a system or its behavior (for example, rotating the coordinate system in which we describe the positions of the particles). Unfortunately, our answers will very often come out backwards, inverted, or otherwise incorrect if we misinterpret active transformations as passive or vice versa. Whenever it matters, when tackling a given problem or examining a certain topic, the best thing to do is to try state explicitly which perspective is being assumed where.

2. Functional Forms versus Function Values

Also note that in physics, by *invariance* sometimes we mean that the value of some function or observable quantity is unchanged, while in other cases we mean that the functional *form* of some functions or differential equations are unchanged, although specific values or solutions, respectively, may change. The latter is sometimes described as a *covariance* rather than invariance, although beware that these various term have several meanings.

3. Continuous versus Discrete Symmetries

Another very important distinction is that between *continuous* and *discrete* transformations. For example, translations and rotations are continuous because we can translate by arbitrarily small shifts or rotate by arbitrarily small angles, and because these transformations can be thought of as being close to close others or "deformed" continuously into them. But mirror reflections are discrete because we either perform the reflection in a given plane or not — there is no way to perform a fraction of the reflection, or a meaningful way to speak of a very small reflection, or continuously vary the reflection. Both types of transformations and the associated symmetries are very important in physics, but discrete symmetries tend to play a less important role in classical mechanics than in quantum mechanics, so except for a few words we will focus here on continuous symmetries.

4. Symmetry Groups

Whether continuous or discrete, transformations associated with symmetries always come together in groups that mathematicians conveniently call "groups." Such groups of symmetry transformation, leaving the system invariant in some specified sense, will have the following properties:

1. "doing nothing" leaves the system invariant in any sense, so is always a trivial symmetry transformation;
2. if one transformation leaves the system invariant, as does a second transformation, then the combined transformation consisting of first one then the other will obviously also do so, so is also a symmetry;
3. obviously, enacting one transformation followed by a second, and then followed by a third, is the same as enacting the first, and then the second followed by the third;
4. if a transformation leaves the system invariant, then the transformation consisting of un-doing or inverting the first will leave the system invariant;
5. first performing then un-doing a symmetry transformation is of course equivalent to doing nothing.

These properties define a *group* of symmetry transformations. Any subset of transformations which itself satisfies all of these conditions, and for which combinations of any of these transformations does not lead to any others outside the subset, is known as a *subgroup*. Group theory is in part, the mathematical description of symmetry.

We say a symmetry group is *generated* by some subset if any symmetry transformation can be thought of as the concatenation of (possibly repeated) transformations from among the set of generators. In physics, the most important discrete symmetry group is generated by just three types of generalized "flips" or "inversions:" *parity*, or spatial reflection of configurations and trajectories through the origin; *charge conjugation*, which replaces the occurrence of any particle with its anti-particle; and *time-reversal*, which reverses the direction of all particle trajectories.

For continuous groups, the generators will depend on continuously-variable real parameters specifying exactly how much translation, or rotation, etc. in some given direction is to be performed. For many groups of interest to physics, these parameters are typically chosen such that as their values get closer to zero, the transformations do less and less, that is, and resemble more closely the identity transformation. Full symmetry groups may involve several independent parameters, although we shall that normally we can restrict attention to sub-groups each depending on only one parameter. Some continuous symmetry group may include sub-groups which are not continuously connected to the identity, but rather some other discrete transformation, like parity inversion or time reversal.

C. Physical Symmetries and Conserved Quantities

Issues of symmetry in our attempts to describe and understand the physical world, and what happens or seems to happen in it, go back, like most of Western thought, at least to the ancient Greek civilization, when various philosophers disputed and debated whether *change* was fundamental and inevitable or superficial and contingent, or whether it was the existence of change or lack of change that needed explanation, or to what extent the particular, concrete, transient, visible material world could be explained by un-changing, abstract, absolute, and necessary universal principles.

Many of these questions have persisted. Exact opinion toward notions of symmetry depended in part on where one fell in the Newton/Leibniz debate as to whether space and time are fundamentally relational or absolute. Some physicists still tend to regard symmetries as something that in a sense, nature must work hard to preserve. Other physicists tend toward a contrary perspective, in which symmetries are precisely those things about which nature is indifferent. Either way, uncovering and exploiting those symmetries will be useful.

1. *Symmetries in Classical Mechanics*

In classical mechanics, sometimes the concept of symmetry applies usefully at the level of a single object — for example, determining the gravitational field from a fixed, spherically-symmetric mass distribution. But in dynamics we are primarily concerned with the evolution of systems over time, and rarely if ever do individual configurations of systems, or else individual trajectories of systems, exhibit fully the kinds of rotational, translational, or other symmetry in which we are interested.

This is usually because initial conditions pick out individual trajectories which break the full symmetry. So to assess symmetry in dynamical settings, we really need to look not at individual trajectories but rather at:

1. invariance in the *form* of the *equations-of-motion* individual trajectories satisfy;
2. invariance of the entire set of trajectories satisfied by the equations-of-motion, starting from all possible initial conditions; or
3. invariance of the *action* which generates the equations-of-motion, if such a variational formulation exists.

Each of these perspectives has its uses, but one reason that the Lagrangian/action description is so useful is that we can assess symmetry at the level of just one function, rather that in terms an infinite set of (usually several) functions over all time, or of an entire system of coupled differential equations.

2. *Infinitesimal, Near-Identity Symmetry Transformations*

In physics, continuous symmetries of interest are not only, well, continuous, but differentiable in a well-defined sense. In fact, we can in effect imagine Taylor expanding such transformations with respect to a group parameter, say ϵ. In such an expansion, the $O(1)$ term typically represents the identity transformation, reflecting the fact that for small ϵ the transformation only shifts things a little bit. The term linear in ϵ represents the leading non-trivial action of the transformation, whereas for sufficiently small ϵ we can neglect the terms proportional to ϵ^2 or higher-order terms.

For ϵ regarded as arbitrarily small, the result is called an infinitesimal, near-identity transformation. (A better terminology would be identity-plus-infinitesimal transformation, but no matter). Such expansion yield infinitesimal transformations from finite ones. For continuous groups depending on one parameter, we can move in the opposite direction, and think generating continuous but finite transformations by concatenating an infinite number of infinitesimal ones.

One of the very nice things about Noether's Theorem is that we can restrict attention to symmetries under just such infinitesimal, near-identity transformations. Usually this simplifies our work significantly, since we can in effect ignore all but two terms in an infinite series. Near-identity transformations must remain invertible for sufficiently small ϵ. In addition symmetry transformations do not commute generically; that is, if we have two transformations, applying the first then the second is not in general the same thing as applying the second then the first. But this lack of commutativity does not arise for infinitesimal transformations, and it is for this reason that we will be able to look separately at each one-parameter symmetry group rather than groups involving several continuous parameters — for example, rotations around a single axis in the limit of small rotation angles, instead of the larger group consisting of all possible spatial rotations.

For Lagrangian systems, we shall consider infinitesimal symmetries which fundamentally act on the generalized coordinates, and possibly on the evolution time parameter. The behavior under the transformation of the generalized velocities cannot be chosen independently, but is inferred by the chain rule, after insisting that transformed generalized velocities remain total transformed-time derivatives of the corresponding transformed generalized coordinates.

3. And Just What Is a Constant of the Motion?

When the numerical value of a certain observable remains constant in time as a physical system evolves, regardless of the initial conditions, then we say in various contexts that observable is *conserved* under the evolution, or is a *conserved quantity*, or is a *dynamical invariant*, or is a *constant of the motion*, or is a *first integral* of the motion. For our purposes we can use any of these terms interchangeably.

Note that although the value assumed by the dynamical invariant does not change in time along any one trajectory, it will generally depend on the initial conditions, that is, on which trajectory the system happens to find itself. (At least, if it does not, the invariant is too trivial to be particularly useful).

Also note the function whose value is conserved under the time evolution may have explicit time dependence (which is then balanced by implicit time variation of other terms), but a dynamical invariant is often more useful if it is a function only of the system variables and not explicitly of the evolution time. On the other hand, even a time-dependent dynamical invariant can be quite useful if it can be evaluated without knowing the full solution. Of course, given an invertible classical system, formally we can always define time-dependent invariants as follows: given the time t and all particle's positions and momenta, output the unique positions and momentum at some "initial" time t_1. While possible in principle, this is rarely helpful in practice unless we already know the solutions to the equations-of-motion.

4. An Example: Kepler's Laws

As a simple example familiar to you from introductory physics, let us consider the case of a planet orbiting a much more massive star, as described by Kepler's Laws derived in turn from Newton's Laws of Motion and Universal Gravitation, assuming for simplicity that both bodies are point masses and that the center-of-mass coincides with the position of the massive star to a good approximation.

Note that both kinetic energy and potential energy are invariant under simultaneous translations of the star and planet by any amount in any direction, and are invariant under rotations of the planet/star system by any angle around any axis. Also note that the energy depends only on the instantaneous positions and velocities of the planet and star, and not explicitly on some absolute time. Additional, somewhat less obvious symmetries also exist, which we will mention below and which Hal may discuss later in the course.

Essentially none of these symmetries are immediately apparent in *any one allowed orbit*, in which relative to the star the planet travels with variable speed along an ellipse, with the star fixed at one foci, while the center-of-mass of the combined system remains either fixed or moves at constant velocity in a given inertial frame. The plane of the ellipse, its eccentricity, the location of it foci and orientation of its axes all remain fixed.

That is, the ellipse traced out by the planet does not rotate, expand, contract, distort, or precess. But if we shift the ellipse (and star and planet) in space by any fixed amount, or rotate by any amount around any axis, or else shift the planet to a new position and corresponding velocity along the orbit, we obtain other trajectories, equally valid apart from the initial conditions which picked our our specific orbit.

Yet this is *not* to imply that the symmetries do not play an important role in the behavior of individual trajectories. It is a deeply beautiful, or ironic, fact (depending on your point-of-view) that any one orbit manifestly breaks the symmetries, and consists of a fixed ellipse of fixed shape in a fixed plane, precisely because of the symmetries in the equations-of-motion, or equivalently in the Lagrangian or action.

D. The Bottom Line

Both symmetries and conservation laws are important in part because they simplify our description and modeling of physical systems. In a sense both reduce the complexity of the problem, and constrain the kind of behaviors we can observe or not. Both involve invariance — in the former case, invariance under symmetry transformations applied to the system or our description of it; in the latter, invariance in time of the value of certain observables as the system evolves. What is perhaps not immediately obvious is that these two seemingly distinct forms of invariance are in fact interrelated.

The previous example of planetary motion is typical of a general feature of Lagrangian dynamical systems — symmetry under a certain continuous group of transformations at the level of the Lagrangian or the equations-of-motion inevitably leads to trajectories which themselves may not be individually symmetric, but along which which certain quantities are necessarily fixed in time.

We explore these connections in more detail next.

II. DYNAMICAL INVARIANTS FROM DYNAMICAL SYMMETRIES

Here we will examine these connections between symmetries and conservation laws from the point of view of forces, equations-of-motion, Lagrangians, and actions, through both examples and some general results.

A. Some Mathematical Notation and Assumptions

Bold-faced Latin or Greek letters will denote multi-component quantities collected into column vectors. In particular, we shall denote a collection of N so-called generalized positions or generalized coordinates by $\boldsymbol{q} = \big(q_1, \dots, q_N\big)^{\mathrm{T}}$, and refer either to a particular point on a trajectory at time t, or the function of time specifying such a trajectory, by $\boldsymbol{q}(t)$ (Which is implied will hopefully be clear depending on the context). Often to work with more compact expressions we will drop various arguments of functions if the meaning is clear. Over-dots will denote total time derivatives, so that the generalized velocities are denoted by $\dot{\boldsymbol{q}}(t) = \frac{d}{dt}\boldsymbol{q}(t) = \big(\dot{q}_1(t), \dots, \dot{q}_N(t)\big)^{\mathrm{T}}$.

Partial derivatives are denoted in the usual shorthand as well, for example

$$\frac{\partial f}{\partial \boldsymbol{q}} = \left(\frac{\partial f}{\partial q_1}, \dots, \frac{\partial f}{\partial q_N}\right)^{\mathrm{T}}, \tag{1}$$

Either we will explicitly include the arguments of which f is a function, or it will hopefully be clear form context what other variables are being held fixed in the partial differentiation. SImilarly, $\frac{\partial^2 f}{\partial \dot{\boldsymbol{q}} \partial \boldsymbol{q}}$ is a matrix of second partial derivatives, etc. When confusion might otherwise arise, we will use expressions like $\mathring{f}$ to denote a total derivatives with respect to a transformed time variable.

In expressions like $\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}} = \frac{\partial L}{\partial \boldsymbol{q}}$, recall that in taking the partial derivatives, we regard the generalized coordinates and the generalized velocities all as independent variables of which the Lagrangian is a function. A partial derivative with respect to one of these variables is performed while holding fixed all the others. *Then* we formally evaluate (or imagine evaluating) the expression for a trajectory $\boldsymbol{q}(t)$, substituting $\boldsymbol{q}(t)$ for $\boldsymbol{q}$ and $\frac{d}{dt}\boldsymbol{q}(t)$ for $\dot{\boldsymbol{q}}$, *and then* we take the remaining total time derivative. All this is a bit glossed over in our usual shorthand notation, and can cause some confusion.

Even if the $\boldsymbol{q}$ do *not* denote Cartesian components of some position vector, we will still use the usual shorthand to denote a *Euclidean* inner product between such N-dimensional vectors, for example,

$$\boldsymbol{q} \cdot \tilde{\boldsymbol{q}} = \sum_{i=1}^{N} q_i \, \tilde{q}_i. \tag{2}$$

This notation will allow us to avoid writing a lot if sums explicitly, but can lead to misunderstanding. Very often we will deal with systems consisting of n particles or other sub-systems. A useful compact notation to partition the components is then the "direct-sum" representation, for example,

$$\boldsymbol{q} = \bigoplus_{j=1}^{n} \boldsymbol{x}_j, \tag{3}$$

where

$$\boldsymbol{q} \cdot \tilde{\boldsymbol{q}} = \sum_j \boldsymbol{x}_j \cdot \tilde{\boldsymbol{x}}_j = \sum_j \sum_\ell x_{j_\ell}\, \tilde{x}_{j_\ell}, \tag{4}$$

and where $\boldsymbol{x}_j$ denotes the generalized coordinates (often just the actual spatial position) of the jth sub-system. Whether a given instance of a dot product is over the whole space of generalized coordinates or over a single-particle's coordinates should be clear from context, but only in the case of Cartesian components is it the usual rotationally-invariant inner product for three-dimensional vectors.

We consider only Lagrangians over a finite number of degrees-of-freedom which depend on the generalized coordinates, possibly explicitly on time, and at most on the *first* derivative in time of the generalized coordinates, i.e., functions of the form $L\big(\boldsymbol{q}, \dot{\boldsymbol{q}}, t\big)$. We will be a little sloppy and often use the same notation to denote the function where all its arguments are regarded as independent variables, or the its value along some trajectory, as a function of time.

We shall everywhere assume that any functions we introduce may be differentiated as many times as needed.

B. Ignorable Coordinates

The Euler-Lagrange equations,

$$\tfrac{d}{dt}\tfrac{\partial L}{\partial \dot{\boldsymbol{q}}} = \tfrac{\partial L}{\partial \boldsymbol{q}} \tag{5}$$

state that the time rate-of-change of the *generalized momenta* $\boldsymbol{p} \equiv \frac{\partial L}{\partial \dot{\boldsymbol{q}}}$ conjugate to the generalized coordinates $\boldsymbol{q}$ are equal the *generalized forces* $\frac{\partial L}{\partial \boldsymbol{q}}$.

A particular generalized coordinate q_i is usually termed *cyclic*, *kinosthenic*, or *ignorable* if it does not actually appear in the Lagrangian. (At one time these terms had slightly different meanings according to some authors, but now are used more or less interchangeably. The rather unilluminating term "cyclic" arose in the context of so-called *action-angle* representations, where the ignorable coordinate could indeed be thought of, at least abstractly, as an angle associated with a uniform rotation in a suitable space.) Anyway, in the case of an ignorable coordinate the corresponding generalized force obviously vanishes identically, so the conjugate (corresponding) momentum vanishes:

$$\tfrac{d}{dt} p_i = \tfrac{dL}{d\dot{q}_i} = \tfrac{dL}{dq_i} = 0. \tag{6}$$

But an equivalent way of saying that $\frac{\partial L}{\partial q_i} = 0$ identically (that is, whatever the values of the generalized coordinates and velocities) is that the Lagrangian L is invariant under infinitesimal shifts in the generalized coordinate q_i, starting from any generalized position, so we immediately we see our first direct connection between an infinitesimal symmetry and a conserved quantity.

Obviously, everything else being equal, it can be useful to choose generalized coordinates so that as many as possible are ignorable. For a single conserved quantity associated with a one-parameter family of infinitesimal symmetries (e.g., translations in one fixed direction, or rotation about one fixed axis), it is in fact always possible in principle, if not necessarily simple in practice, to choose generalized coordinates where this one dynamical invariant emerges as a momentum conjugate to an ignorable coordinate (that does not appear in the transformed Lagrangian). It may not be possible to do this simultaneously for multiple invariants.

Although the Lagrangian formulation gives us great freedom to choose generalized coordinates, it is also not necessarily convenient to use up this freedom in finding ignorable coordinates. Our goal in the remainder of these notes is in effect to generalize and expand on this connection between symmetries and conservation in cases where the symmetry is not so obvious as an ignorable coordinate explicitly missing from a Lagrangian. Before we construct the full apparatus of Noether's Theorem, we will look at some familiar cases.

C. Canonical Linear Momentum from Translation Invariance of the Lagrangian

We can generalize what we mean by invariance under infinitesimal transformations to cases where it is not necessarily just a matter of of a shift in one generalized coordinate.

As a simple but commonly-encountered example, consider an isolated system of n classical particles moving in three dimensional space with a potential energy of the form $U(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n, t)$ that is invariant under simultaneous translations of all Cartesian positions $\boldsymbol{x}_j$, $j = 1, \ldots, n$ by arbitrary shifts proportional to in some fixed displacement vector $\boldsymbol{u}$:

$$U(\boldsymbol{x}_1 + \epsilon\boldsymbol{u}, \ldots, \boldsymbol{x}_n + \epsilon\boldsymbol{u}, t) = U(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N, t) \tag{7}$$

for any sufficiently small choice of real parameter ϵ, and where again $\boldsymbol{u} \in \mathbb{R}^3$ is a fixed, time-independent vector specifying the common direction of spatial displacement in three-dimensional space for all particle positions.

Note that under the translations $\boldsymbol{x}_j \to \boldsymbol{x}_j + \epsilon\,\boldsymbol{u}$, the corresponding velocities are unchanged: $\dot{\boldsymbol{x}}_j \to \dot{\boldsymbol{x}}_j + \frac{d}{dt}\left[\epsilon\,\boldsymbol{u}\right] = \dot{\boldsymbol{x}}_j + \boldsymbol{0} = \dot{\boldsymbol{x}}_j$. Hence the usual kinetic energy,

$$K = K(\dot{\boldsymbol{x}}_1, \ldots, \dot{\boldsymbol{x}}_n) = \sum_j \tfrac{1}{2} m_j |\dot{\boldsymbol{x}}_j|^2 \tag{8}$$

is invariant under such translations, and hence the Lagrangian $L = K - U$ is also invariant:

$$L(\boldsymbol{x}_1 + \epsilon\boldsymbol{u}, \ldots, \boldsymbol{x}_n + \epsilon\boldsymbol{u}, \dot{\boldsymbol{x}}_1 \ldots, \dot{\boldsymbol{x}}_n, t) = L(\boldsymbol{x}_1 \ldots, \boldsymbol{x}_n, \dot{\boldsymbol{x}}_1 \ldots, \dot{\boldsymbol{x}}_N, t). \tag{9}$$

Now, differentiating both sides of this last relation with respect to ϵ, and then evaluating at $\epsilon = 0$ and for some trajectories $\boldsymbol{x}_j(t)$ satisfying the equations-of-motion, we find

$$\tfrac{\partial}{\partial \epsilon} L\,\Big|_{\epsilon=0} = \sum_j \boldsymbol{u} \cdot \tfrac{\partial L}{\partial \boldsymbol{x}_j} = 0, \tag{10}$$

and using the Euler-Lagrange equations $\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{x}}_j} = \frac{\partial L}{\partial \boldsymbol{x}_j}$, and the fact that $\boldsymbol{u} = |\boldsymbol{u}|\,\hat{\boldsymbol{u}}$ is assumed time-independent, we infer

$$\tfrac{d}{dt}\left[\hat{\boldsymbol{u}} \cdot \sum_j \tfrac{\partial L}{\partial \dot{\boldsymbol{x}}_j}\right] = 0, \tag{11}$$

or in other words: the component of the total linear momentum $\boldsymbol{P} = \sum\limits_j \frac{\partial L}{\partial \dot{\boldsymbol{x}}_j}$ in the direction of $\hat{\boldsymbol{u}}$ is conserved under the time evolution. If this holds for three linearly-independent choices of $\boldsymbol{u}$, then it must hold for all components of the momentum, and we can conclude the total linear momentum $\boldsymbol{P}$ is conserved.

Note in the particular case where the $\boldsymbol{x}_j$ represent Cartesian coordinates of positions, and the kinetic energy is given by the standard form (8), the total momentum is just $\boldsymbol{P} = \sum\limits_j \boldsymbol{p}_j$, where $\boldsymbol{p}_j = \frac{\partial L}{\partial \dot{\boldsymbol{x}}_j} = m\dot{\boldsymbol{x}}_j$, as would be expected.

If, in particular, the potential energy is of the form of additive pairwise interactions between particles that are a function only of distance, i.e., if

$$U(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n, t) = \sum_{j<k} \Phi_{jk}\big(|\boldsymbol{x}_j - \boldsymbol{x}_k|, t\big), \tag{12}$$

then (all components of) the total linear momentum $\boldsymbol{P}$ must be conserved, even though momentum may be *exchanged* between particles due to their interactions. Notice that the potential energy is *not* invariant if we shift just one particle's coordinates, so individual particle momenta $\boldsymbol{p}_j$ need not be conserved.

However, if in addition, *external* potential energy terms that break translation invariance are present, for example if

$$U(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n, t) = \sum_{j} \Psi_j(\boldsymbol{x}_j, t) + \sum_{j<k} \Phi_{jk}\big(|\boldsymbol{x}_j - \boldsymbol{x}_k|, t\big), \tag{13}$$

where at least some $\Psi_j(\boldsymbol{x}, t)$ are non-constant, then the total linear momentum of the system is no longer conserved. In effect, this is because the additional terms can be seen to represent interactions with an "external world" which can exchange momentum with the system of interest.

D. Energy Conservation from Time Independence of the Lagrangian

Instead of looking at infinitesimal shifts in the generalized coordinates, we can also consider the consequences of infinitesimal shifts in any *explicit* time dependence in the Lagrangian. Suppose the Lagrangian does not depend *explicitly* on time at any fixed values of the generalized coordinates and generalized velocities. This means

$$\tfrac{\partial}{\partial t} L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = 0 \qquad \text{for all } t, \tag{14}$$

or equivalently that

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t + \epsilon\tau) = L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \qquad \text{for all } t \text{ and } \epsilon, \tag{15}$$

given some choice of the fixed time offset τ. These are to hold at any prescribed values of $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$.

But if instead we consider the value of the Lagrangian along a trajectory, differentiate with respect to time, and invoke the Euler-Lagrange equations, we find

$$\tfrac{d}{dt} L\big(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t\big) = \dot{\boldsymbol{q}}(t)\cdot\tfrac{\partial L}{\partial \boldsymbol{q}} + \ddot{\boldsymbol{q}}(t)\cdot\tfrac{\partial}{\partial \dot{\boldsymbol{q}}} + \tfrac{\partial L}{\partial t} L = \dot{\boldsymbol{q}}(t)\cdot\tfrac{d}{dt}\tfrac{\partial}{\partial \dot{\boldsymbol{q}}} + \ddot{\boldsymbol{q}}(t)\cdot\tfrac{\partial L}{\partial \dot{\boldsymbol{q}}} + \tfrac{\partial L}{\partial t}. \tag{16}$$

Using the product rule, this becomes

$$\tfrac{d}{dt} L = \tfrac{d}{dt}\big[\dot{\boldsymbol{q}}\cdot\tfrac{\partial L}{\partial \dot{\boldsymbol{q}}}\big] + \tfrac{\partial L}{\partial t}, \tag{17}$$

or after rearrangement,

$$\tfrac{d}{dt}\big[\dot{\boldsymbol{q}}\cdot\tfrac{\partial L}{\partial \dot{\boldsymbol{q}}} - L\big] = -\tfrac{\partial L}{\partial t} \tag{18}$$

along the trajectory.

1. *Conservation of the Hamiltonian*

So if in fact $\frac{\partial L}{\partial t} = 0$ (everywhere and always), then

$$\tfrac{d}{dt}\big[\dot{\boldsymbol{q}}\cdot\tfrac{\partial L}{\partial \dot{\boldsymbol{q}}} - L\big] = 0, \tag{19}$$

and we obtain a conservation law associated with this invariance in the Lagrangian.

The quantity in the brackets, $H \equiv \dot{\boldsymbol{q}}\cdot\frac{\partial L}{\partial \dot{\boldsymbol{q}}} - L$, is sufficiently important to earn its own name: the *Hamiltonian*, and abbreviation, usually H. (The Hamiltonian is more naturally regarded as a function of the generalized coordinates and their conjugate momenta, rather than the generalized coordinates and velocities, but that is a story for later). The Hamiltonian is related to the Lagrangian and its derivatives in a way known as a *Legendre Transformation*, which you will learn more about later in this course as well as in other physics courses like Statistical Mechanics.

2. *When Can the Hamiltonian Be Identified with the Total Energy?*

At least in most cases where the Lagrangian L has no explicit time dependence (and in other cases as well) this conserved Hamiltonian H can be identified with the system's *total energy.* In particular, when the Lagrangian is given by $L = K(\dot{\boldsymbol{q}}) - U(\boldsymbol{q})$, while the energy $E = K(\boldsymbol{q}, \dot{\boldsymbol{q}}) + U(\boldsymbol{q})$, we see that we can identify H with E provided

$$H = \dot{\boldsymbol{q}}\cdot\frac{\partial K}{\partial \dot{\boldsymbol{q}}} - (K - U) = E = K + U, \tag{20}$$

or

$$\dot{\boldsymbol{q}}\cdot\frac{\partial K}{\partial \dot{\boldsymbol{q}}} = 2\,K \tag{21}$$

Remembering Euler's Homogeneous Function Theorem, we realize that as long as the kinetic energy $K(\dot{\boldsymbol{q}})$ is smooth, this condition can hold if and only if the kinetic energy K is a homogenous function of degree 2 of the generalized velocities $\dot{\boldsymbol{q}}$:

$$K(\boldsymbol{q}, \lambda\dot{\boldsymbol{q}}) = \lambda^2\,K(\boldsymbol{q}, \dot{\boldsymbol{q}}) \qquad \text{for all real, positive } \lambda. \tag{22}$$

Note this is true for the usual expression for kinetic energy of a system of particles (expressed in Cartesian coordinates), $K = \sum_j m_j|\dot{\boldsymbol{x}}_j|^2$, as well as for the rotational kinetic energy of a rigid body, $K = \sum_j\sum_j \mathcal{I}_{jk}\,\omega_j\omega_k$, where the $\mathcal{I}_{jk}$ are the Cartesian components of the moment-of-inertia tensor, and ω_j the components of angular velocity.

More generally, this homogeneity requirement holds in any case that the kinetic energy can be expressed as a quadratic form in the generalized velocities, possibly with coefficients that depend on the coordinates.

This particular, but commonly-encountered, form for the kinetic energy is preserved under any invertible, time-independent and velocity-independent coordinate transformations. Suppose the generalized coordinates $\boldsymbol{q}$ are related to another set of coordinates $\boldsymbol{r}$ by *time-independent*, invertible transformations:

$$\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{r}), \tag{23a}$$

$$\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{q}), \tag{23b}$$

where $\boldsymbol{q}\big(\boldsymbol{r}(\boldsymbol{q}')\big) = \boldsymbol{q}'$ and $\boldsymbol{r}\big(\boldsymbol{q}(\boldsymbol{r}')\big) = \boldsymbol{r}'$, such that the kinetic energy is a quadratic form with respect to the generalized velocities $\dot{\boldsymbol{r}}$:

$$K\big(\boldsymbol{r}, \dot{\boldsymbol{r}}\big) = \tfrac{1}{2}\dot{\boldsymbol{r}}^{\mathrm{T}}\mathcal{M}\big(\boldsymbol{r}\big)\,\dot{\boldsymbol{r}}, \tag{24}$$

for some symmetric matrix $\mathcal{M} = \mathcal{M}(\boldsymbol{r}) = \mathcal{M}^{\mathrm{T}}$ which may be a function of the generalized coordinates $\boldsymbol{r}$, but not the generalized velocities $\dot{\boldsymbol{r}}$. In the two coordinate systems, the generalized velocities are related by

$$\dot{\boldsymbol{q}} = \frac{\partial \boldsymbol{q}}{\partial \boldsymbol{r}}\,\dot{\boldsymbol{r}}, \tag{25a}$$

$$\dot{\boldsymbol{r}} = \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{q}}\,\dot{\boldsymbol{q}}, \tag{25b}$$

where $\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{q}} = \big(\frac{\partial \boldsymbol{q}}{\partial \boldsymbol{r}}\big)^{-1}$, and and thus the kinetic energy can be re-expressed as

$$K(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \tfrac{1}{2}\dot{\boldsymbol{q}}^{\mathrm{T}}\Big[\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{q}}^{\mathrm{T}}\mathcal{M}\big(\boldsymbol{r}(\boldsymbol{q})\big)\,\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{q}}\Big]\,\dot{\boldsymbol{q}}, \tag{26}$$

which is a quadratic form with respect to the transformed generalized velocities $\dot{\boldsymbol{q}}$. Notice that the matrix $\left[\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{q}}^{\mathrm{T}} \mathcal{M}\big(\boldsymbol{r}(\boldsymbol{q})\big) \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{q}}\right]$ is symmetric, and will be positive definite (and hence invertible) if $\mathcal{M}$ is positive definite.

3. A Caveat

Ambiguous statement's like "the laws of nature do not care when we start or clocks" can lead to some misunderstanding as to exactly what form of invariance is needed to establish energy conservation.

While we have used the derivative $\frac{d}{dt} L\big(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t\big)$ in our proof of energy conservation, we are *not* assuming a symmetry of the form $L\big(\boldsymbol{q}(t+\epsilon\tau), \dot{\boldsymbol{q}}(t+\epsilon\tau), t+\epsilon\tau\big) = L\big(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t\big)$. While in a few systems (notably, free particles) the value of Lagrangian happens to be conserved under time evolution in this way, the symmetry associated with energy involves shifts in the *explicit* time dependence of the Lagrangian, at *fixed* values of the generalized coordinates and velocities.

The shift of the time parameter *everywhere*, that is both the explicit and implicit time dependence of the Lagrangian evaluated on a trajectory, represents in effect merely a change in our book-keeping and is a symmetry of sorts in any dynamical system, albeit a rather trivial one which would not in generalized to any interesting conservation laws.

E. Angular Momentum Conservation from Rotational Invariance of the Lagrangian

Now that you have the idea, I will actually leave it to the reader to demonstrate conservation of total angular momentum $\boldsymbol{J} = \sum_j \boldsymbol{x}_j \times \boldsymbol{p}_j$ directly form the *rotational* invariance of the Lagrangian. The proof is similar to the case of linear momentum. If you get stuck, do not worry; we will derive this conservation law directly from Noether's Theorem below.

F. Galilean Invariance and Center-of-Mass Motion

In Newtonian (that is, pre-relativistic) theories, recall that the laws of nature are expected to be the same in all inertial frames related by any one of *ten* types of transformations: all spatial translations, all spatial rotations, time translations, or *Galilean boosts* by any velocity. The latter means that one inertial frame is moving at a fixed velocity $\boldsymbol{V}$ relative to the other. Under a Galilean boost, the corresponding coordinates are related by

$$t' = t, \tag{27a}$$

$$\boldsymbol{x}_j' = \boldsymbol{x}_j - \boldsymbol{V} t, \tag{27b}$$

in a *passive* sense if the primed frame is moving at $\boldsymbol{V}$ relative to the unprimed frame but otherwise has its coordinate axes parallel to the corresponding unprimed axes. The corresponding velocities then transform as

$$\dot{\boldsymbol{x}}_j' = \dot{\boldsymbol{x}}_j - \boldsymbol{V}. \tag{28}$$

Consider a Lagrangian for n classical particles, of the form $L = K - U$, where once again the kinetic energy is of form (8) and the potential energy is of the form

$$U(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n, t) = \sum_{j<k} \mathcal{U}_{jk}\big(\boldsymbol{x}_j - \boldsymbol{x}_k, t\big). \tag{29}$$

Invariance under Galilean boost is easiest to see at the level of the equations-of-motion. Because $\boldsymbol{V}$ is constant and the potential energy terms depend only on relative positions, under a boost it is obvious that both the accelerations $\ddot{\boldsymbol{x}}_j$ and forces $-\sum_k \boldsymbol{\nabla}\mathcal{U}_{jk}\big(\boldsymbol{x}_j - \boldsymbol{x}_k, t\big)$ are left unchanged.

However, let us look more closely at the behavior of the Lagrangian. Under an *active*, infinitesimal Galilean boost by $\epsilon\boldsymbol{V}$, the potential energy U remains invariant, while the kinetic energy becomes

$$K(\dot{\boldsymbol{x}}_1 + \epsilon\boldsymbol{V}, \ldots, \dot{\boldsymbol{x}}_n + \epsilon\boldsymbol{V}) = K(\dot{\boldsymbol{x}}_1, \ldots, \dot{\boldsymbol{x}}_n) + \epsilon\, M\boldsymbol{V}\cdot \tfrac{d}{dt}\boldsymbol{X}_{\rm cm} + \epsilon^2\, \tfrac{1}{2}M|\boldsymbol{V}|^2, \tag{30}$$

where $M = \sum_j = m_j$ is the total mass,

$$\boldsymbol{X}_{\rm cm} = \tfrac{1}{M}\sum_j m_j \boldsymbol{x}_j \tag{31}$$

is the center-of-mass, and

$$\tfrac{d}{dt}\boldsymbol{X}_{\rm cm} = \boldsymbol{V}_{\rm cm} = \tfrac{1}{M}\sum_j m_j \dot{\boldsymbol{x}}_j = \tfrac{\boldsymbol{P}}{M} \tag{32}$$

is the center-of-mass velocity. So on the one hand

$$\tfrac{\partial}{\partial\epsilon} L(, \boldsymbol{x}_1 + \epsilon\boldsymbol{V}t, \ldots, \boldsymbol{x}_n + \epsilon\boldsymbol{V}t, \dot{\boldsymbol{x}}_1 + \epsilon\boldsymbol{V}, \ldots, \dot{\boldsymbol{x}}_n + \epsilon\boldsymbol{V}, t)\big|_{\epsilon=0} = M\boldsymbol{V}\cdot \tfrac{d}{dt}\boldsymbol{X}_{\rm cm}. \tag{33}$$

On the other hand, a formal use of the chain rule and Euler-Lagrange equations yields

$$\begin{aligned} \tfrac{d}{d\epsilon} L(\boldsymbol{x}_1 + \epsilon\boldsymbol{V}t, \ldots, \boldsymbol{x}_n + \epsilon\boldsymbol{V}t, \dot{\boldsymbol{x}}_1 + \epsilon\boldsymbol{V}, \ldots, \dot{\boldsymbol{x}}_n + \epsilon\boldsymbol{V}, t)\big|_{\epsilon=0} &= \sum_j \big[\boldsymbol{V}t \cdot \tfrac{\partial L}{\partial \boldsymbol{x}_j} + \boldsymbol{V}\cdot \tfrac{\partial L}{\partial \dot{\boldsymbol{x}}_j}\big] \\ &= \sum_j \big[\boldsymbol{V}t\cdot \tfrac{d}{dt}\boldsymbol{p}_j + \boldsymbol{V}\cdot \boldsymbol{p}_j\big] = \boldsymbol{V}t \cdot \tfrac{d}{dt}\boldsymbol{P} + \boldsymbol{V}\cdot\boldsymbol{P}. \end{aligned} \tag{34}$$

Equating these two expressions and rearranging, we have

$$M\boldsymbol{V}\cdot \tfrac{d}{dt}\boldsymbol{X}_{\rm cm} = \boldsymbol{V}t\cdot \tfrac{d}{dt}\boldsymbol{P} + \boldsymbol{V}\cdot\boldsymbol{P}, \tag{35}$$

and from the product rule and the fact that $\boldsymbol{V}$ is a constant vector, we find $\tfrac{d}{dt}\boldsymbol{V}\cdot\boldsymbol{X}_{\rm cm} = \tfrac{d}{dt}\big[\boldsymbol{V}\cdot \tfrac{\boldsymbol{P}}{M}\, t\big]$, or equivalently:

$$\tfrac{d}{dt}\hat{\boldsymbol{V}}\cdot\big[\boldsymbol{X}_{\rm cm} - \tfrac{\boldsymbol{P}}{M}\, t\big] = 0. \tag{36}$$

This is the time-dependent constant-of-the-motion associated with invariance under a boost by $\boldsymbol{V}$. If this holds for arbitrary boost directions, then we may conclude

$$\tfrac{d}{dt}\big[\boldsymbol{X}_{\rm cm} - \tfrac{\boldsymbol{P}}{M}\, t\big] = \boldsymbol{0}, \tag{37}$$

and in fact

$$\boldsymbol{X}_{\rm cm}(t) = \boldsymbol{X}_{\rm cm}(0) + \tfrac{\boldsymbol{P}}{M}\, t. \tag{38}$$

Notice that we did not actually use the fact that the total momentum $\boldsymbol{P}$ is conserved, or the fact that $\boldsymbol{P} = M\boldsymbol{V}_{\rm cm}$, although both are true in this case by virtue of the presumed form for k and U. But in general it is important to note that spatial translations by a fixed displacement, and Galilean boosts by a fixed velocity, are actually distinct symmetries.

III. NOETHER'S THEOREM

Some common strategies and patterns are beginning to emerge, but in order to develop the connection between symmetries and conservations laws for Lagrangian systems more systematically, we shall work up to a proof of Noether's (First) Theorem, and offer some discussion and examples.

There are various versions of Noether's Theorem, in varying degrees of generality. We will proof here the following: given a system with a finite number of degrees-of-freedom, and described by a second-order Lagrangian, if the action is suitably invariant under a one-parameter group of infinitesimal, near-identity transformations of the generalized coordinates and/or the evolution time, then there must exist a corresponding dynamical invariant whose value is conserved along all allowed trajectories.

We will not go into detail here, but it turns out the proof works backwards, and so the converse is also true: any conservation law of a certain form will be associated with a dynamical symmetry.

A. Biographical Aside: Emmy Noether

Amalie Emmy Noether (1882–1935) has been considered by Einstein and others, and continues to be considered, to be the greatest woman mathematician in history. (Incidentally, opinions as to the greatest male mathematician of all time are divided. Popular candidates are Archimedes, Newton, Euler, and Gauss). She had to suffer many obstacles to produce great mathematics. As a female, she was denied entrance into the gymnasium, or college preparatory school, in her native Germany. Despite support from David Hilbert, Felix Klein and other influential mathematicians of her day, she went years without official academic positions due to entrenched sexism in the German universities. Of Jewish heritage, she then had to flee Germany with the rise of the Nazi party, and then died young in America from complications from a tumor. While most of her work was in abstract algebra, she is most famous in the mathematical physics for one of the central results of Lagrangian dynamics, now known as *Noether's Theorem.*

B. Symmetry Transformations and Action Invariance

In its most basic version studied here, Noether's theorem relates conservation laws for a finite-degree-of-freedom Lagrangian system to action invariance under a one-parameter group of *infinitesimal*, *near-identity*, *smooth* symmetry transformations of the generalized coordinates and/or the time parameter. The induced transformation of the generalized velocities are then fixed by assuming the generalized velocities remain interpreted as the total time derivatives of the generalized coordinates. Every such one-parameter group of symmetries will be associated with its own conservation law.

Again, working with the infinitesimal versions is generally much simpler. Although occasionally invariance under finite transformations is obvious — for example, we can confirm at a glance the rotational symmetry of an expression like $\frac{1}{2}m_j|\dot{\boldsymbol{x}}_j|^2$, with infinitesimal transformations we end up being able to do more differentiation (mechanical, and relatively simple) rather than integration (more difficult), or avoid systems of nonlinear algebraic equations (also hard), and along the way need only worry about the leading terms in Taylor expansions.

Because the transformations are infinitesimally close to the "do-nothing" identity transformations, again note that such transformations will necessarily remain invertible when ϵ is regarded as sufficiently small. Therefore such transformations can be regarded either in a passive sense, as well-behaved if ever-so-slight changes to the coordinates describing the trajectories, or in an active sense, as infinitesimal perturbations to the physical trajectories. Both interpretations will have their uses, as long as we keep them straight.

First we should review how the Lagrangian itself transforms under a *passive* change of coordinates of the form $\boldsymbol{q}' = \boldsymbol{q}'(\boldsymbol{q}, t)$, implying $\mathring{\boldsymbol{q}}' = \frac{\partial}{\partial t}\boldsymbol{q}' + \dot{\boldsymbol{q}} \cdot \frac{\partial}{\partial \boldsymbol{q}}\boldsymbol{q}'$. In Newtonian (pre-relativistic) physics, it is often said that the "Lagrangian transforms as a scalar," meaning

$$L'(\boldsymbol{q}', \mathring{\boldsymbol{q}}', t) = L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \tag{39}$$

which is to say, under a (passive) change of coordinates, the functional form of the Lagrangian changes so that its value at any instant in a given dynamical state does not change, regardless of which mathematical coordinates are used to describe it. In this way the Euler-Lagrange equations derived from the Lagrangians will generate the same physical trajectories in either coordinate system.

Notice that it is the value of the Lagrangian, not its actual functional form, which is kept fixed. In general, it would not even make sense to try to keep the form fixed — for example, if we transformed from Cartesian coordinates x, y, z to spherical coordinates, ρ, θ, ϕ, we cannot just stick the second transformed coordinate (i.e., the polar angle θ) into the argument slot for the second original coordinate (i.e., Cartesian component y) in the original Lagrangian, for they do not serve the same role and do not even have the same units.

However, we can adopt a still more expansive view of what counts as a legitimate transformation in our description of trajectories, and of what corresponding change to the Lagrangian leads to equivalent dynamical equations. So far, we have not messed with time. However, we may for example wish to introduce an overall shift in time (i.e., change in origin from which we measure elapsed time) or an overall scaling (i.e., a change in the units in which we measure time). Actually, we can go further and accommodate virtually any smooth, invertible mappings from the original generalized coordinates $\boldsymbol{q}$ and original time t to transformed coordinates $\boldsymbol{q}'$ and transformed time t':

$$t' = t'(\boldsymbol{q}, t), \tag{40a}$$

$$\boldsymbol{q}' = \boldsymbol{q}'(\boldsymbol{q}, t), \tag{40b}$$

from which we can infer the corresponding transformation rule for the generalized velocities:

$$\mathring{\boldsymbol{q}}' \equiv \frac{d}{dt'}\boldsymbol{q}' = \frac{\frac{d\boldsymbol{q}'}{dt}}{\frac{dt'}{dt}} = \frac{\frac{\partial}{\partial t}\boldsymbol{q}' + \dot{\boldsymbol{q}}\cdot\frac{\partial}{\partial \boldsymbol{q}}\boldsymbol{q}'}{\frac{\partial}{\partial t}t' + \dot{\boldsymbol{q}}\cdot\frac{\partial}{\partial \boldsymbol{q}}t'}. \tag{41}$$

Note that we are assuming that the coordinate and temporal mappings do *not* depend on the generalized velocities, ensuring that the Euler-Lagrange equations remain second-order in either coordinate system.

To determine how the Lagrangian must in turn transform, we first consider the action functional, since integrals behave rather simply under a change of integration variable. *Hamilton's Principle* tells us that the allowed trajectories are those for which the variation δS in the action is at least of second-order when we imagine making first-order infinitesimal variations in the trajectories, keeping fixed the generalized positions at the fixed initial and final times.

In order to ensure that the action remains stationary in the new coordinates, the value of the action should be invariant, except possibly for terms depending on the generalized coordinates and time at the *endpoints* of the trajectory, which are then unaffected by the variation considered in applying Hamilton's Principle.

That is, we must require:

$$S' = S + F\big(\boldsymbol{q}(t_2), t_2\big) - F\big(\boldsymbol{q}(t_1), t_1\big) \tag{42}$$

for some function $F(\boldsymbol{q}, t)$ whose values at the initial and final points along the trajectory provide any extra boundary terms that may emerge under the coordinate transformations, and which we stress again is a function only of quantities that are fixed at the endpoints under the variations allowed by Hamilton's Principle. Clearly $\delta S' = \delta S$ under such infinitesimal variations,and so Hamilton's Principle will pick out the same physical trajectories in either coordinate system, just described differently.

Writing the various terms in (42) as definite integrals, we find

$$\int_{t_1'}^{t_2'} dt'\, L'\big(\boldsymbol{q}'(t'), \mathring{\boldsymbol{q}}'(t), t'\big) = \int_{t_1}^{t_2} dt\, L\big(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t'), t\big) + \int_{t_1}^{t_2} dt\, \tfrac{d}{dt} F\big(\boldsymbol{q}(t), t\big), \tag{43}$$

where we have introduced the natural shorthand $t_2' = t'\big(\boldsymbol{q}(t_2), t_2\big)$ and $t_1' = t'\big(\boldsymbol{q}(t_1), t_1\big)$. Transforming the integration variable on the left-hand side, rearranging using linearity on the right, and suppressing some complicated nested functional dependence to avoid notational clutter, we have

$$\int_{t_1}^{t_2} dt\, \tfrac{dt'}{dt}\, L'\big(\boldsymbol{q}', \mathring{\boldsymbol{q}}', t'\big) = \int_{t_1}^{t_2} dt\, \Big[L\big(\boldsymbol{q}, \dot{\boldsymbol{q}}, t\big) + \tfrac{d}{dt} F\big(\boldsymbol{q}, t\big)\Big], \tag{44}$$

where both integrals are now performed with respect to the original time t. Because t_1 and t_2 are essentially arbitrary, it must be the case that the integrands are in fact equal:

$$L'(\boldsymbol{q}', \mathring{\boldsymbol{q}}', t')\, \tfrac{dt'}{dt} = L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) + \tfrac{d}{dt} F(\boldsymbol{q}, t), \tag{45}$$

which establishes the desired transformation rule. The addition of the $\frac{d}{dt}F$ term is known for historical reasons as a *Lagrangian gauge transformation*, and $F(\boldsymbol{q}, t)$ itself as a gauge generator or gauge function.

2. When Are Two Lagrangians Equivalent?

The possibility of a gauge transformations arises because although a well-defined Lagrangian uniquely generates a set of Euler-Lagrange equations, the equations-of-motion do not uniquely determine a Lagrangian. Notice that if we use the identity transformation $t' = t$ and $\boldsymbol{q}' = \boldsymbol{q}$, the transformation rule (45) becomes simply

$$L'(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) + \tfrac{d}{dt} F(\boldsymbol{q}, t). \tag{46}$$

Clearly this relation holds true if $F(\boldsymbol{q}, t)$ is a constant function, but it is in fact true more generally. Indeed it is straightforward to work backwards, and use the Fundamental Theorem of Calculus to verify that if two Lagrangians (which are functions of the same set of generalized coordinates and velocities, and time) differ by a total time derivative, then they generate the same equations-of-motion. The Lagrangians are then said to be *gauge-equivalent* or sometimes just equivalent. To ensure that L' depends at most only on the first derivatives of $\boldsymbol{q}(t)$, note that F can depend on $\boldsymbol{q}$ but not $\dot{\boldsymbol{q}}$.

In terms of the corresponding action functionals, we have

$$\begin{aligned} S' &= \int_{t_1}^{t_2} dt\, L' = \int_{t_1}^{t_2} dt\, \big(L + \tfrac{dF}{dt}\big) = \int_{t_1}^{t_2} dt\, L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) + \int_{t_1}^{t_2} dt\, \tfrac{d}{dt} F(\boldsymbol{q}, t) \\ &= S + \big[F(\boldsymbol{q}_2, t_2) - F(\boldsymbol{q}_1, t_1)\big]. \end{aligned} \tag{47}$$

Thus we conclude that $\delta S' = \delta S$ under the sort of variations considered in Hamilton's Principle with the endpoints fixed in time and generalized position, and therefore these two actions are necessarily stationary for exactly the same functions $\boldsymbol{q}(t)$ satisfying these boundary conditions.

The reader may also find it comforting if a bit tedious to verify directly from the Euler-Lagrange equations that gauge-equivalent Lagrangians indeed do lead to the same equations-of-motion. A proof is presented in the Appendix A, which if nothing else confirms that a proof using the action functional is simpler.

Interestingly, gauge-equivalence in this sense is a sufficient but not necessary condition for two Lagrangians to generate the same dynamical equations. There do exist inequivalent Lagrangians in the sense that they do not differ by a total time derivative, yet do generate the same equations-of-motion. We offer it as a ***Take-Home Challenge Problem*** for the reader find an example of such Lagrangians. (HINT: think about what you have just studied, namely simple harmonic oscillators.). A solution is described in the Appendix. However, these sorts of exceptional cases do not seem to play a direct role in the understanding of conservation laws.

3. Lagrangian Transformations Under General Symmetry Transformations

So far our transformation rule (45) tells us how the Lagrangian itself must transform under any smooth, invertible, but otherwise arbitrary coordinate/time transformations. The recipe tells us to change the form of the Lagrangian function so as to preserve the value of the action, up to boundary terms.

We cannot in general consider preserving a functional *form* of the Lagrangian for arbitrary transformations. But if the transformations can also be meaningfully interpreted in an *active* sense, specifically as a shift of one physical trajectory into another, described before and after by the same general type of coordinates (in particular, with the same units), then we could "legally" insert the transformed variables into the old Lagrangian, or the old variables into the new Lagrangian.

In fact, if the transformation is additionally a symmetry, then the transformed Lagrangian should retain the same *functional form* before and after the transformation; that is, $L'(\boldsymbol{q}, \mathring{\boldsymbol{q}}', t') = L(\boldsymbol{q}, \mathring{\boldsymbol{q}}', t')$ as functions, and we obtain a precise mathematical statement of this dynamical symmetry:

$$L(\boldsymbol{q}', \mathring{\boldsymbol{q}}', t')\, \tfrac{dt'}{dt} = L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) + \tfrac{d}{dt} F(\boldsymbol{q}, t), \tag{48}$$

for some gauge function $F(\boldsymbol{q}, t)$.

The simplest case arises if $t' = t$ and $F(\boldsymbol{q}, t)$ is simply constant, in which case the value of the Lagrangian is unchanged, namely $L(\boldsymbol{q}', \mathring{\boldsymbol{q}}', t) = L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$, but the notion of dynamical symmetries associated with conservation laws is somewhat more general.

4. Lagrangian Transformations under Infinitesimal Symmetry Transformations

To treat infinitesimal, near-identity symmetry transformations, we will parameterize the transformations by a real, dimensionless parameter ϵ, and essentially restrict attention to arbitrarily small values of ϵ, taking the limits as $\epsilon \to 0$ at the *end* of the calculation if needed. Along the way we can generally ignore terms that are $O(\epsilon^2)$ or of higher order.

Specifically, we can write the transformations for time and the generalized coordinates in the form:

$$T = T(\boldsymbol{q}, t; \epsilon) = t + \epsilon\, \tau(\boldsymbol{q}, t) + O(\epsilon^2), \tag{49a}$$

$$\boldsymbol{Q} = \boldsymbol{Q}(\boldsymbol{q}, t; \epsilon) = \boldsymbol{q} + \epsilon\, \boldsymbol{\xi}(\boldsymbol{q}, t) + O(\epsilon^2), \tag{49b}$$

where the corresponding rule for transforming generalized velocities becomes

$$\mathring{\boldsymbol{Q}} = \tfrac{d}{dT}\boldsymbol{Q} = \frac{\frac{d\boldsymbol{Q}}{dt}}{\frac{dT}{dt}} = \frac{\dot{\boldsymbol{q}} + \epsilon\, \dot{\boldsymbol{\xi}}}{1 + \epsilon\, \dot{\tau}} = \dot{\boldsymbol{q}} + \epsilon\, (\dot{\boldsymbol{\xi}} - \dot{\tau}\, \dot{\boldsymbol{q}}) + O(\epsilon^2), \tag{50}$$

and the total time derivatives of $\tau(\boldsymbol{q}, t)$ and $\boldsymbol{\xi}(\boldsymbol{q}, t)$ are to be calculated by the chain rule in the familiar way. For $\epsilon = 0$ the transformations clearly reduce to the identity transformation.

Since the coordinate transformations are parameterized by ϵ, so too will be the transformed Lagrangian and the gauge generator $F(\boldsymbol{q}, t)$. Setting $F(\boldsymbol{q}, t) \equiv f_0(\boldsymbol{q}, t) + \epsilon\, f(\boldsymbol{q}, t) + \epsilon^2\, f_2(\boldsymbol{q}, t) + \ldots$, and expanding the symmetry condition (48) in powers of ϵ using the chain rule, we find

$$L + \epsilon\, \dot{\tau} L + \epsilon\, \tau\, \tfrac{\partial L}{\partial t} + \epsilon\, \boldsymbol{\xi} \cdot \tfrac{\partial L}{\partial \boldsymbol{q}} + \epsilon\, \big(\dot{\boldsymbol{\xi}} - \dot{\tau}\, \dot{\boldsymbol{q}}\big) \cdot \tfrac{\partial L}{\partial \dot{\boldsymbol{q}}} + O(\epsilon^2) = L + \epsilon \dot{f} + O(\epsilon^2), \tag{51}$$

where everything is now evaluated using the original coordinates and time. Equating powers of ϵ, we arrive at our expression for symmetry under the infinitesimal transformation (49):

$$\dot{\tau}\, L + \tau\, \tfrac{\partial L}{\partial t} + \boldsymbol{\xi} \cdot \tfrac{\partial L}{\partial \boldsymbol{q}} + \big(\dot{\boldsymbol{\xi}} - \dot{\tau}\, \dot{\boldsymbol{q}}\big) \cdot \tfrac{\partial L}{\partial \dot{\boldsymbol{q}}} - \dot{f} = 0 \tag{52}$$

for some function $f(\boldsymbol{q}, t)$. The symmetry condition includes a certain directional derivative of the Lagrangian, but also terms due to the integration measure and the possible gauge transformation.

C. A Derivative-Based Proof

Now we can finally offer a proof of Noether's Theorem, in more generality than is offered in your textbook . Again, various versions of the theorem (and the converse) are discussed in the literature. Here we aim to demonstrate that if the action is invariant under a group of infinitesimal, near-identity, invertible transformations depending on time and the generalized positions (but no derivatives), then there exists a dynamical invariant, associated with the symmetry in question, conserved along any trajectories satisfying the original Euler-Lagrange equations.

First, we offer a "direct assault" by manipulating various derivatives. The proof is straightforward but perhaps not entirely illuminating. Our strategy is to start from (**??**), add and subtract canceling quantities, rearrange, and simplify. These additional terms will be designed to allow us either to (a) combine various terms into total derivatives using the product rule, or (b) cancel combinations of terms using the Euler-Lagrange equations.

So, starting with,

$$\dot{\tau}\, L + \tau\, \tfrac{\partial L}{\partial t} + \boldsymbol{\xi}\cdot \tfrac{\partial L}{\partial \boldsymbol{q}} + \big(\dot{\boldsymbol{\xi}} - \dot{\tau}\, \dot{\boldsymbol{q}}\big)\cdot \tfrac{\partial L}{\partial \dot{\boldsymbol{q}}} - \dot{f} = 0 \tag{53}$$

we can write

$$\dot{\tau}\, L + \tau\, \tfrac{dL}{dt} - \tau\left[\dot{\boldsymbol{q}}\cdot \tfrac{\partial L}{\partial \boldsymbol{q}} + \ddot{\boldsymbol{q}}\cdot \tfrac{\partial L}{\partial \dot{\boldsymbol{q}}}\right] + \boldsymbol{\xi}\cdot \tfrac{\partial L}{\partial \boldsymbol{q}} + \big(\dot{\boldsymbol{\xi}} - \dot{\tau}\, \dot{\boldsymbol{q}}\big)\cdot \tfrac{\partial L}{\partial \dot{\boldsymbol{q}}} - \dot{f} = 0, \tag{54}$$

or

$$\dot{\tau}\, L + \tau\, \dot{L} - \dot{f} + \big(\boldsymbol{\xi} - \tau\, \dot{\boldsymbol{q}}\big)\cdot \tfrac{\partial L}{\partial \boldsymbol{q}} + \big(\dot{\boldsymbol{\xi}} - \dot{\tau}\, \dot{\boldsymbol{q}} - \tau\, \ddot{\boldsymbol{q}}\big)\cdot \tfrac{\partial L}{\partial \dot{\boldsymbol{q}}} = 0. \tag{55}$$

Adding and subtracting $\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}}$, we obtain

$$\dot{\tau}\, L + \tau\, \dot{L} - \dot{f} + \big(\boldsymbol{\xi} - \tau\, \dot{\boldsymbol{q}}\big)\cdot\left[\tfrac{\partial L}{\partial \boldsymbol{q}} - \tfrac{d}{dt}\tfrac{\partial L}{\partial \dot{\boldsymbol{q}}}\right] + \big(\boldsymbol{\xi} - \tau\, \dot{\boldsymbol{q}}\big)\cdot \tfrac{d}{dt}\tfrac{\partial L}{\partial \dot{\boldsymbol{q}}} + \big(\dot{\boldsymbol{\xi}} - \dot{\tau}\, \dot{\boldsymbol{q}} - \tau\, \ddot{\boldsymbol{q}}\big)\cdot \tfrac{\partial L}{\partial \dot{\boldsymbol{q}}} = 0. \tag{56}$$

Combining the first two terms together, and also the last two terms together, using the Product Rule, we find

$$\tfrac{d}{dt}\big[\tau\, L\big] - \tfrac{d}{dt} f + \big(\boldsymbol{\xi} - \tau\, \dot{\boldsymbol{q}}\big)\cdot\left[\tfrac{\partial L}{\partial \boldsymbol{q}} - \tfrac{d}{dt}\tfrac{\partial L}{\partial \dot{\boldsymbol{q}}}\right] + \tfrac{d}{dt}\left[\big(\boldsymbol{\xi} - \tau\, \dot{\boldsymbol{q}}\big)\cdot \tfrac{\partial L}{\partial \dot{\boldsymbol{q}}}\right] = 0, \tag{57}$$

and finally combining various total time derivatives together using linearity, we have

$$\tfrac{d}{dt}\left[\tau\, L + \big(\boldsymbol{\xi} - \tau\, \dot{\boldsymbol{q}}\big)\cdot \tfrac{\partial L}{\partial \dot{\boldsymbol{q}}} - f\right] + \big(\boldsymbol{\xi} - \tau\, \dot{\boldsymbol{q}}\big)\cdot\left[\tfrac{\partial L}{\partial \boldsymbol{q}} - \tfrac{d}{dt}\tfrac{\partial L}{\partial \dot{\boldsymbol{q}}}\right] = 0, \tag{58}$$

so if the symmetry condition (49) holds, then along a trajectory satisfying the Euler-Lagrange equations, the observable $\mathcal{J}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = \left[\tau\, L + \big(\boldsymbol{\xi} - \tau\, \dot{\boldsymbol{q}}\big)\cdot \frac{\partial L}{\partial \dot{\boldsymbol{q}}} - f\right]$ is conserved.

D. An Action-Oriented Proof

We can also prove Noether's Theorem using our action functional, which to my taste better reveals the connection between the conservation law and our original variational principle.

We define the difference in the transformed versus original actions:

$$\Delta S(\epsilon) \equiv \int_{T_1}^{T_2} dT\, L\big(\boldsymbol{Q}(T;\epsilon), \mathring{\boldsymbol{Q}}(T;\epsilon), T\big) - \int_{t_1}^{t_2} dt\, L\big(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t\big), \tag{59}$$

along some prescribed and fixed trajectory $\boldsymbol{q}(t)$. Here $T_2 \equiv T(\boldsymbol{q}_2, t_2; \epsilon)$ and $T_1 \equiv T(\boldsymbol{q}_1, t_1; \epsilon)$, where $\boldsymbol{q}_2 = \boldsymbol{q}(t_2)$ and $\boldsymbol{q}_1 = \boldsymbol{q}(t_1)$, and the transformed trajectory $\boldsymbol{Q}(T;\epsilon)$ is defined by

$$\boldsymbol{Q}(T;\epsilon) \equiv \boldsymbol{q}\big(t(T;\epsilon)\big) + \epsilon\, \boldsymbol{\xi}\Big(\boldsymbol{q}\big(t(T;\epsilon)\big), t(T;\epsilon)\Big), \tag{60}$$

in which $t(T;\epsilon)$ is the inverse function at fixed ϵ of

$$T(t;\epsilon) \equiv t + \epsilon\, \tau\Big(\boldsymbol{q}\big(t(T;\epsilon)\big), t(T;\epsilon)\Big); \tag{61}$$

while $\mathring{\boldsymbol{Q}}(T;\epsilon) \equiv \frac{\partial}{\partial T}\boldsymbol{Q}(T;\epsilon)$.

On the one hand, if the coordinate/temporal transformations (49) do represent (to $O(\epsilon)$) a dynamical symmetry, then by integrating (48), or equivalently from simply noting that (actively) transformed trajectories must also be physically valid ones under Hamilton's Principle for a symmetry, we see that

$$\Delta S(\epsilon) = \epsilon\, f\big(\boldsymbol{q}(t), t\big)\,\Big|_{t_1}^{t_2} + O(\epsilon^2). \tag{62}$$

On the other hand, we can directly Taylor expand the first integral in (59) about $\epsilon = 0$, assuming a fixed trajectory $\boldsymbol{q}(t)$. We use the Leibniz Integral Rule (see Appendix C) in order to account for leading-order contributions from ϵ-dependence in both the integration limits and the integrand of the first integral:

$$\begin{aligned} \Delta S(\epsilon) = 0 + \epsilon \Big[L(q_2, t_2)\, \tfrac{\partial T(q_2,t_2;\epsilon)}{\partial \epsilon}\big|_{\epsilon=0} - L(q_1,t_1)\, \tfrac{\partial T(q_1,t_1;\epsilon)}{\partial\epsilon}\big|_{\epsilon=0} \Big] \\ + \epsilon \lim_{\epsilon\to 0} \int_{T_1}^{T_2} dT\, \Big[\big(\tfrac{\partial \boldsymbol{Q}}{\partial\epsilon}\big)_T \cdot \tfrac{\partial L}{\partial \boldsymbol{Q}} + \big(\tfrac{\partial \mathring{\boldsymbol{Q}}}{\partial\epsilon}\big)_T \cdot \tfrac{\partial L}{\partial \mathring{\boldsymbol{Q}}} \Big] \;+\; O(\epsilon^2). \end{aligned} \tag{63}$$

In evaluating the contributions from the boundary terms, the derivatives are taken at fixed values of the initial and final times and positions as expressed in the original coordinates., and then evaluated at $\epsilon = 0$. However, note that in evaluating the contribution from the first integrand, we have to take the indicated derivatives not at a fixed value of the time t, but rather at a fixed values of the integration variable T, but need retain only $O(1)$ terms since the $O(\epsilon)$ or higher-order terms will vanish when ultimately evaluated as $\epsilon \to 0$. Using various chain rules where necessary, these derivatives are given by

$$\tfrac{\partial T(q_1,t_1;\epsilon)}{\partial\epsilon}\big|_{\epsilon=0} = \tau(\boldsymbol{q}_1, t_1), \tag{64a}$$

$$\tfrac{\partial T(q_2,t_2;\epsilon)}{\partial\epsilon}\big|_{\epsilon=0} = \tau(\boldsymbol{q}_2, t_2), \tag{64b}$$

$$\big(\tfrac{\partial \boldsymbol{Q}(T;\epsilon)}{\partial\epsilon}\big)_T\big|_{\epsilon=0} = \boldsymbol{\xi}\big(\boldsymbol{q}(t),t\big) - \tau\big(\boldsymbol{q}(t),t\big)\,\dot{\boldsymbol{q}}(t) + O(\epsilon), \tag{64c}$$

$$\big(\tfrac{\partial \mathring{\boldsymbol{Q}}(T;\epsilon)}{\partial\epsilon}\big)_T\big|_{\epsilon=0} = \dot{\boldsymbol{\xi}}\big(\boldsymbol{q}(t),t\big) - \dot{\tau}\big(\boldsymbol{q}(t),t\big)\,\ddot{\boldsymbol{q}}(t) - \tau\big(\boldsymbol{q}(t),t\big)\,\ddot{\boldsymbol{q}}(t) + O(\epsilon). \tag{64d}$$

(64e)

Substituting in these expressions and evaluating the indicated terms as $\epsilon \to 0$, we have

$$\Delta S(\epsilon) = \epsilon\,\tau\, L\,\big|_{t_1}^{t_2} + \epsilon \int_{t_1}^{t_2} dt\, \big[(\boldsymbol{\xi} - \tau\,\dot{\boldsymbol{q}}) \cdot \tfrac{\partial L}{\partial \boldsymbol{q}} + (\dot{\boldsymbol{\xi}} - \tau\,\ddot{\boldsymbol{q}} - \dot{\tau}\,\ddot{\boldsymbol{q}}) \cdot \tfrac{\partial L}{\partial \dot{\boldsymbol{q}}} \big] \;+\; O(\epsilon^2), \tag{65}$$

and noting that $(\dot{\boldsymbol{\xi}} - \tau\,\ddot{\boldsymbol{q}} - \dot{\tau}\,\ddot{\boldsymbol{q}}) = \frac{d}{dt}\big[\boldsymbol{\xi} - \tau\,\dot{\boldsymbol{q}}\big]$, we can integrate-by-parts the second term in the integrand, rearrange, and obtain:

$$\Delta S(\epsilon) = \epsilon \Big[\tau\, L + (\boldsymbol{\xi} - \tau\,\dot{\boldsymbol{q}}) \cdot \tfrac{\partial L}{\partial \dot{\boldsymbol{q}}} \Big]\Big|_{t_1}^{t_2} + \epsilon \int_{t_1}^{t_2} dt\, \big(\boldsymbol{\xi} - \tau\,\dot{\boldsymbol{q}}\big) \cdot \big[\tfrac{\partial L}{\partial \boldsymbol{q}} - \tfrac{d}{dr}\tfrac{\partial L}{\partial \dot{\boldsymbol{q}}} \big] \;+\; O(\epsilon^2). \tag{66}$$

It may not look like we have accomplished much, but this is essentially our main result, and in fact shows how both Noether's theorem and the Euler-Lagrange equations naturally follow from a single variational expression.

To reproduce the equations-of-motion, we: leave unchanged the time parameter t, setting $\tau(\boldsymbol{q}, t) \equiv 0$, demand that $\boldsymbol{\eta}(\boldsymbol{q},\, t) = \boldsymbol{\eta}(t)$ is an integrable function of time only, satisfying $\boldsymbol{\eta}(t_1) = \boldsymbol{\eta}(t_2) = \boldsymbol{0}$ but is otherwise otherwise arbitrary, and require that $\Delta S(\epsilon)$ vanish up to $O(\epsilon)$ inclusive. Under these conditions, the boundary terms in (66) all vanish (either because τ vanishes identically or else because

$\boldsymbol{\eta}(t)$ vanishes at the initial and final times), so in fact $\Delta S = \delta S$, exactly the variation appearing in Hamilton's Principle, and we have

$$\Delta S(\epsilon) = \delta S = 0 + \epsilon \int_{t_1}^{t_2} dt\, \boldsymbol{\eta}(t) \cdot \left[\tfrac{\partial L}{\partial \boldsymbol{q}} - \tfrac{d}{dr} \tfrac{\partial L}{\partial \dot{\boldsymbol{q}}} \right] + O(\epsilon^2) = 0 + 0 + O(\epsilon^2) = O(\epsilon^2). \tag{67}$$

Therefore the overall $O(\epsilon)$ term must vanish:

$$\int_{t_1}^{t_2} dt\, \boldsymbol{\eta}(t) \cdot \left[\tfrac{\partial L}{\partial \boldsymbol{q}} - \tfrac{d}{dr} \tfrac{\partial L}{\partial \dot{\boldsymbol{q}}} \right] = 0, \tag{68}$$

and since $\boldsymbol{\eta}(t)$ can be adjusted arbitrarily on the interior of the integration interval, we infer that the integrand itself must vanish at (almost) all moments between the initial and final times:

$$\tfrac{\partial L}{\partial \boldsymbol{q}} - \tfrac{d}{dt} \tfrac{\partial L}{\partial \dot{\boldsymbol{q}}} = 0. \tag{69}$$

The resulting differential equations are of course just the by-now familiar Euler-Lagrange Equations governing the time evolution. In fact here we just reproduced the original argument leading to the Euler-Lagrange equations.

Instead, in order to see the connection to conserved quantities, let us assume that the trajectory $\boldsymbol{q}(t)$ satisfies the Euler-Lagrange equations, while the action satisfies the symmetry conditions.

Because $\boldsymbol{q}(t)$ satisfies the Euler-Lagrange equations for $t_1 < t < t_2$, the integrand, and hence the integral, in (66) both vanish. Equating (62) and (66), we obtain

$$\epsilon\, f(\boldsymbol{q}(t), t) \Big|_{t_1}^{t_2} + O(\epsilon^2) = \Delta S(\epsilon) = \epsilon \left[\tau\, L + (\boldsymbol{\xi} - \tau\, \dot{\boldsymbol{q}}) \cdot \tfrac{\partial L}{\partial \dot{\boldsymbol{q}}} \right] \Big|_{t_1}^{t_2} + O(\epsilon^2), \tag{70}$$

and by equating the $O(\epsilon)$ terms, we deduce

$$f(\boldsymbol{q}(t), t) \Big|_{t_1}^{t_2} = \left[\tau\, L + (\boldsymbol{\xi} - \tau\, \dot{\boldsymbol{q}}) \cdot \tfrac{\partial L}{\partial \dot{\boldsymbol{q}}} \right] \Big|_{t_1}^{t_2}, \tag{71}$$

or equivalently

$$\left[\tau\, L + (\boldsymbol{\xi} - \tau\, \dot{\boldsymbol{q}}) \cdot \tfrac{\partial L}{\partial \dot{\boldsymbol{q}}} - f \right] \Big|_{t_1}^{t_2} = 0, \tag{72}$$

indicating that the sum of terms inside the brackets assumes the same value at the initial and final times. Because we are in principle free to choose the final time t_2 arbitrarily, we can in fact set $t_2 = t$, and this implies that the value of the sum of terms inside the brackets is constant throughout the time evolution:

$$\left[\tau\, L + (\boldsymbol{\xi} - \tau\, \dot{\boldsymbol{q}}) \cdot \tfrac{\partial L}{\partial \dot{\boldsymbol{q}}} - f \right] \Big|_{t_1}^{t} = 0, \tag{73}$$

or equivalently

$$\tfrac{d}{dt} \left[\tau\, L + (\boldsymbol{\xi} - \tau\, \dot{\boldsymbol{q}}) \cdot \tfrac{\partial L}{\partial \dot{\boldsymbol{q}}} - f \right] = \tfrac{d}{dt} \mathcal{J} = 0 \tag{74}$$

along an actual trajectory, exactly the same conservation law established above.

Notice that in general, the dynamical invariant $\mathcal{J}(\boldsymbol{q}, t)$ associated with a symmetry may have *explicit* time dependence entering through any of the contributing terms. Also notice that, as we would expect, the form of the dynamical invariant depends upon the Lagrangian and its derivatives, as well as the specific infinitesimal symmetry under which the action is invariant.

IV. SOME APPLICATIONS OF NOETHER'S THEOREM

We can use Noether's Theorem to establish directly from the corresponding symmetries several well-known conservation laws, including in particular those we worked out somewhat more laboriously above.

A. Canonical Linear Momentum and Translation Invariance

Consider a system described by a Lagrangian $L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ that is invariant under identical infinitesimal shifts in all the generalized positions $\boldsymbol{q}$ at any time t.

To determine the form of the infinitesimal transformation of relevance here, note that we are leaving time alone, so that $\tau(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \equiv 0$, and consider infinitesimal translations in the generalized coordinates along a fixed direction, such that

$$\boldsymbol{q} \to \boldsymbol{Q} = \boldsymbol{q} + \epsilon\, \boldsymbol{\xi}_0, \tag{75}$$

so that $\boldsymbol{\xi}(\boldsymbol{q}, t) \equiv \boldsymbol{\xi}_0$ is in in fact just a constant vector. In this case the generalized velocities are unchanged:

$$\dot{\boldsymbol{q}} \to \mathring{\boldsymbol{Q}} = \dot{\boldsymbol{q}}. \tag{76}$$

If under this infinitesimal displacement, the symmetry condition (52) holds without a gauge term, i.e., if it is the case that

$$\dot{\tau}\, L + \tau\, \tfrac{\partial L}{\partial t} + \boldsymbol{\xi}\cdot \tfrac{\partial L}{\partial \boldsymbol{q}} + \big(\dot{\boldsymbol{\xi}} - \dot{\tau}\, \dot{\boldsymbol{q}}\big)\cdot \tfrac{\partial L}{\partial \dot{\boldsymbol{q}}} - \dot{f} = 0 + 0 + \boldsymbol{\xi}_0\cdot \tfrac{\partial L}{\partial \boldsymbol{q}} + 0 + 0 = \boldsymbol{\xi}_0\cdot \tfrac{\partial L}{\partial \boldsymbol{q}} = 0, \tag{77}$$

then Noether's theorem (74) immediately yields

$$\tfrac{d}{dt}\big[0 + (\boldsymbol{\xi}_0 - \boldsymbol{0})\cdot \tfrac{\partial L}{\partial \dot{\boldsymbol{q}}} - 0\big] = \tfrac{d}{dt}\big[\boldsymbol{\xi}_0\cdot \tfrac{\partial L}{\partial \dot{\boldsymbol{q}}}\big] = 0, \tag{78}$$

for any $\boldsymbol{\xi}_0$ for which the infinitesimal translation invariance condition holds.

An important special case:

In a typical case, the generalized positions might correspond to $\boldsymbol{q} = \bigoplus_j \boldsymbol{x}_j$, where $\boldsymbol{x}_j$ for $j = 1, \ldots, N$, are the actual three-dimensional spatial positions of N particles with the usual kinetic energy K and a potential energy U, whereas $\boldsymbol{\xi}_0 = \bigoplus_j \boldsymbol{u}$ will correspond to fixed parallel, and in fact equal, displacements $\boldsymbol{u} \in \mathbb{R}^3$ of the spatial positions for each particle.

Each particle's contribution to the kinetic energy is obviously invariant under any translation, so translation invariance of L is equivalent to

$$\boldsymbol{\xi}_0 \cdot \tfrac{\partial L}{\partial \boldsymbol{q}} = -\boldsymbol{u}\cdot \sum_j \tfrac{\partial U}{\partial \boldsymbol{x}_j} = 0, \tag{79}$$

and if this holds true then in this case the momentum conservation reduces to:

$$\tfrac{d}{dt}\big[\hat{\boldsymbol{u}} \cdot \sum_j \tfrac{\partial L}{\partial \dot{\boldsymbol{x}}_j}\big] = 0. \tag{80}$$

Invariance of the Lagrangian with respect to infinitesimal translations in various directions leads to momentum conservation in those same directions. If the Lagrangian is invariant under infinitesimal translations in three linear independent directions, it must in fact invariant in all directions, and the total linear momentum $\boldsymbol{P} \equiv \sum_j \frac{\partial L}{\partial \dot{\boldsymbol{x}}_j} = \sum_j \boldsymbol{p}_j$ will be conserved.

B. Angular Momentum and Rotational Symmetry

Although we could develop this in somewhat more generality, let us consider a system of N particles with positions $\boldsymbol{x}_j$, $j = 1, \ldots, N$, moving in three-dimensional space, according to a Lagrangian of the form

$$L\big(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N, \dot{\boldsymbol{x}}_1, \ldots, \dot{\boldsymbol{x}}_N, t\big) = K\big(|\dot{\boldsymbol{x}}_1|, \ldots, |\dot{\boldsymbol{x}}_N|\big) - U\big(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N, t\big), \tag{81}$$

and we consider infinitesimal rotations of the particle positions (and hence also velocities).

How do we express an infinitesimal rotation of the usual spatial positions and velocities by an arbitrarily small angle ϵ about some fixed rotation axis $\hat{\boldsymbol{n}}$?

For the positions, such a rotation can be written as

$$\boldsymbol{x}_j \to \boldsymbol{X}_j = \boldsymbol{x}_j + \epsilon\, \hat{\boldsymbol{n}} \times \boldsymbol{x}_j + O(\epsilon^2). \tag{82}$$

If this is not obvious, try convincing yourself as follows: without any real loss of generality, choose coordinate axes so that $\hat{\boldsymbol{n}}$ corresponds to, say, the z axis, so the rotation must leave the values of z coordinates unchanged but rotate the x and y coordinates, keeping the latter in a plane perpendicular to the z axis and maintaining the transverse distance to the z axis at least up to $O(\epsilon)$. Specifically, an infinitesimal rotation about the z axis can be written as $z \to z$, $x \to \cos(\epsilon)\, x - \sin(\epsilon)\, y = x - \epsilon y + O(\epsilon^2)$, and $y \to \cos(\epsilon)\, y + \sin(\epsilon)\, x = y + \epsilon x + O(\epsilon^2)$. It is easy to verify that in coordinate-free language this is equivalent to (82).

By differentiating with respect to time t, we can infer the corresponding transformations for the velocities:

$$\dot{\boldsymbol{x}}_j \to \dot{\boldsymbol{X}}_j = \dot{\boldsymbol{x}}_j + \epsilon\, \hat{\boldsymbol{n}} \times \dot{\boldsymbol{x}}_j + O(\epsilon^2), \tag{83}$$

which reveal, as anticipated, that the velocities rotate in the same way as the spatial positions.

Thus we have $\boldsymbol{\xi} = \bigoplus_j \hat{\boldsymbol{n}} \times \boldsymbol{x}_j$. We still leave the evolution time t unchanged, so that $\tau \equiv 0$. The kinetic energy depends only on the particle speeds, which are obviously invariant under any rotation. In particular,

$$\dot{\boldsymbol{\xi}} \cdot \tfrac{\partial L}{\partial \dot{\boldsymbol{q}}} = \sum_j \tfrac{d}{dt}\big(\hat{\boldsymbol{n}} \times \boldsymbol{x}_j\big) \cdot \tfrac{\partial K}{\partial \dot{\boldsymbol{x}}_j} = \sum_j \big(\hat{\boldsymbol{n}} \times \dot{\boldsymbol{x}}_j\big) \cdot m_j \dot{\boldsymbol{x}}_j = \sum_j \hat{\boldsymbol{n}} \cdot \big(\dot{\boldsymbol{x}}_j \times \dot{\boldsymbol{x}}_j\big) = 0, \tag{84}$$

where in the last steps we used the invariance of a triple product under cyclic permutations, and anti-symmetry of the cross product.

So if the potential energy satisfies

$$\sum_j \big(\hat{\boldsymbol{n}} \times \boldsymbol{x}_j\big) \cdot \tfrac{\partial U}{\partial \boldsymbol{x}_j} = \hat{\boldsymbol{n}} \cdot \sum_j \big(\boldsymbol{x}_j \times \tfrac{\partial U}{\partial \boldsymbol{x}_j}\big) = 0, \tag{85}$$

then the symmetry condition will hold in the absence of a gauge term. In particular, when the potential energy U is of the form (12), for which

$$\tfrac{\partial \Phi_{jk}}{\partial \boldsymbol{x}_j} = \tfrac{\boldsymbol{R}_{jk}}{R_{jk}} \tfrac{\partial \Phi_{jk}}{\partial R_{jk}}, \tag{86}$$

where $\boldsymbol{R}_{jk} \equiv (\boldsymbol{x}_j - \boldsymbol{x}_k) = -\boldsymbol{R}_{kj}$, and $R_{jk} \equiv |\boldsymbol{R}_{jk}| = R_{kj}$, then

$$\boldsymbol{x}_j \times \tfrac{\partial \Phi_{jk}}{\partial \boldsymbol{x}_j} + \boldsymbol{x}_k \times \tfrac{\partial \Phi_{jk}}{\partial \boldsymbol{x}_k} = \big(\boldsymbol{x}_j \times \boldsymbol{R}_{jk} + \boldsymbol{x}_k \times \boldsymbol{R}_{kj}\big) \tfrac{\partial \Phi_{jk}}{\partial R_{jk}} = -\big(\boldsymbol{x}_j \times \boldsymbol{x}_k + \boldsymbol{x}_k \times \boldsymbol{x}_j\big) \tfrac{\partial \Phi_{jk}}{\partial R_{jk}} = \boldsymbol{0}, \tag{87}$$

and summing over all distinct pairs of particles, we see that (86) will be satisfied for any choice of $\hat{\boldsymbol{n}}$.

We can conclude from Noether's Theorem that

$$\tfrac{d}{dt}\Big[\sum_j (\hat{\boldsymbol{n}}\times\boldsymbol{x}_j)\cdot m_j\dot{\boldsymbol{x}}_j\Big] = \tfrac{d}{dt}\,\hat{\boldsymbol{n}}\cdot\Big[\sum_j \boldsymbol{x}_j\times m_j\,\dot{\boldsymbol{x}}_j\Big] = 0, \tag{88}$$

implying conservation of the component of angular momentum in the direction $\hat{\boldsymbol{n}}$ of rotational symmetry, i.e.,

$$\tfrac{d}{dt}\,\hat{\boldsymbol{n}}\cdot\boldsymbol{J} = \boldsymbol{0}, \tag{89}$$

along any orbit, where $\boldsymbol{J}(t) \equiv \sum_j \boldsymbol{x}_j(t)\times m_j\,\dot{\boldsymbol{x}}_j(t) = \sum_j \boldsymbol{x}_j(t)\times\boldsymbol{p}_j(t)$ is the usual total angular momentum defined with respect to the common origin of the position coordinates. If the rotational invariance holds for arbitrary direction $\hat{\boldsymbol{n}}$, as in this case considered here, then the angular momentum is entirely conserved: $\frac{d}{dt}\boldsymbol{J}(t) = 0$.

C. Energy Conservation and Time Translation Symmetry

Suppose instead that we do no not shift the generalized coordinates, but do shift time by constant amount: $t \to T = t + \epsilon\,\tau_0$. So setting $\boldsymbol{\xi} = \boldsymbol{0}$ and $\tau = \tau_0$, we see that if $\frac{\partial L}{\partial t} = 0$, Noether's Theorem implies that

$$\tfrac{d}{dt}\big[\tau_0 L - \tau_0\dot{\boldsymbol{q}}\cdot\tfrac{\partial L}{\partial\dot{\boldsymbol{q}}}\big] = 0, \tag{90}$$

and dividing out by τ_0, we have $\frac{d}{dt}H = \frac{d}{dt}\big[L - \dot{\boldsymbol{q}}\cdot\frac{\partial L}{\partial\dot{\boldsymbol{q}}}\big] = 0$. It is that simple — once we have the machinery in place.

For an isolated physical system, different sub-systems may exchange energy while keeping the total energy fixed. A Lagrangian that includes explicit time dependence indicates that the system of interest is not truly isolated, but is exchanging energy with the external world — albeit in a manner that is not tracked entirely self-consistently from the perspective of both the system and its environment, but is simply prescribed in the time dependence. This is generally a good approximation if the "environment" is so large or so well controlled that the effects of the system on its evolution can be neglected.

D. Symmetry Under Galilean Boosts and Center-of-Mass Motion

So far we have not made any use of the gauge freedom, but it is necessary for the case of boost invariance. For our usual system of otherwise isolated point-particles, consider the transformation $t \to t$, $\boldsymbol{x}_j = \boldsymbol{x}_j + \epsilon\,\boldsymbol{V}\,t$, and therefore $\dot{\boldsymbol{x}}_j \to \dot{\boldsymbol{x}}_j + \epsilon\,\boldsymbol{V}$. This corresponds to $\tau \equiv 0$ and $\boldsymbol{\xi} = \bigoplus \boldsymbol{V}\,t$.

A potential of the form (12) is completely invariant under such an infinitesimal boost, but the kinetic energy of the form (8) picks up additional terms. Altogether,

$$\boldsymbol{\xi}\cdot\frac{\partial}{\partial\boldsymbol{\xi}} + \dot{\boldsymbol{\xi}}\cdot\frac{\partial}{\partial\dot{\boldsymbol{\xi}}} = \sum_j \boldsymbol{V}\cdot\tfrac{\partial K}{\partial\dot{\boldsymbol{x}}_j} = \boldsymbol{V}\cdot\sum_j m_j\,\dot{\boldsymbol{x}}_j = \boldsymbol{V}\cdot\boldsymbol{P} = \tfrac{d}{dt}\big[\boldsymbol{V}\cdot M\boldsymbol{X}_{\text{cm}}\big]. \tag{91}$$

From Noether's Theorem, we deduce the conservation law

$$\tfrac{d}{dt}\Big[\sum_j t\,\boldsymbol{V}\cdot m_j\,\dot{\boldsymbol{x}}_j - \boldsymbol{V}\cdot M\,\boldsymbol{X}_{\text{cm}}\Big] = \tfrac{d}{dt}\,\hat{\boldsymbol{V}}\cdot\Big[t\,\boldsymbol{P} - M\,\boldsymbol{X}_{\text{cm}}\Big] = 0, \tag{92}$$

or equivalently,

$$\tfrac{d}{dt}\,\hat{\boldsymbol{V}}\cdot\Big[\boldsymbol{X}_{\text{cm}} - \tfrac{\boldsymbol{P}}{M}\,t\Big] = 0 \tag{93}$$

for *any* direction $\hat{\boldsymbol{V}}$, which verifies the fact that the center-of-mass moves according to: $\frac{d}{dt}\big[\boldsymbol{X}_{\text{cm}} - \frac{\boldsymbol{P}}{M}\,t\big] = \boldsymbol{0}$, or in other words $\boldsymbol{X}_{\text{cm}}(t) = \boldsymbol{X}_{\text{cm}}(0) + \frac{\boldsymbol{P}(t)}{M}\,t$. For the Lagrangian under consideration here, total momentum $\boldsymbol{P}$ is also conserved because of translational symmetry, so together these conservation laws imply that the center-of-mass moves at a uniform velocity given by $\dot{\boldsymbol{X}}_{\text{cm}}(t) = \dot{\boldsymbol{X}}_{\text{cm}}(0) = \boldsymbol{P}/M$.

E. Additional Examples

In order gain a bit more familiarity, and to convince us that Noether's Theorem have broader applicability than the conservation laws we already deduced, let us look at a few cases where the system is not isolated, and/or where the symmetry is somewhat less obvious. For simplicity, we examine point-particle systems subject to external forces derivable from a few different forms of the potential energy

1. *Particles Subject to a Time-Dependent but Spatially-Uniform External Force*

Suppose the point particles interact according to conservative, internal forces governed by a translation-invariant and time-independent internal potential energy, but are all also subject to external forces $\boldsymbol{F}_j(t)$ which are time-dependent but spatially uniform. The total Lagrangian can be written

$$L = \sum_j \tfrac{1}{2} m_j |\dot{\boldsymbol{x}}_j|^2 - \sum_{j,k} \Phi_{jk}(\boldsymbol{x}_j - \boldsymbol{x}_k) + \sum_j \boldsymbol{x}_j \cdot \boldsymbol{F}_j(t). \tag{94}$$

Overall, we no longer necessarily have complete invariance of the Lagrangian with respect to spatial or temporal translations, or with respect to spatial rotations. However, under the simultaneous parallel spatial translations $\boldsymbol{x}_j \to \boldsymbol{x}_j + \epsilon\boldsymbol{\Delta}$ for any constant vector $\boldsymbol{\Delta}$, the Lagrangian picks up a gauge term,

$$L \to L + \sum_j \epsilon\, \boldsymbol{\Delta} \cdot \boldsymbol{F}_j(t) = \epsilon \tfrac{d}{dt} \Big[\boldsymbol{\Delta} \cdot \sum_j \int_{t_0}^{t} dt'\, \boldsymbol{F}_j(t') \Big], \tag{95}$$

for some (arbitrary) choice of a reference time t_0, and (exploiting the fact that the direction of $\boldsymbol{\Delta}$ is also arbitrary), Noether's Theorem guarantees that

$$\tfrac{d}{dt} \sum_j \Big[m_j \dot{x}_j(t) - \int_{t_0}^{t} dt'\, \boldsymbol{F}_j(t') \Big] = \boldsymbol{0}, \tag{96}$$

which implies (by taking the total time derivatives) that the acceleration of the center-of-mass satisfies a form of Newton's Second Law with respect to the total mass and the total *external* force:

$$M\, \ddot{\boldsymbol{X}}_{\text{cm}}(t) = \dot{\boldsymbol{P}}(t) = \sum_j \boldsymbol{F}_j(t); \tag{97}$$

or (upon integration), that the total change in linear momentum over any time interval will equal the total *external* impulse,

$$\boldsymbol{P}(t_2) - \boldsymbol{P}(t_1) = \int_{t_0}^{t_2} dt' \sum_j \boldsymbol{F}_j(t') - \int_{t_0}^{t_1} dt' \sum_j \boldsymbol{F}_j(t') = \int_{t_1}^{t_2} dt' \sum_j \boldsymbol{F}_j(t'). \tag{98}$$

These relations can of course be derived by more elementary means, by summing over the equations-of-motion for each particle and noting that the internal forces satisfy Newton's Third Law, but they do emerge more or less automatically from Noether's Theorem.

2. A Particle in a Rotating Potential

Suppose a point particle is moving subject to a uniformly-rotating potential, such that in cylindrical coordinates, the Lagrangian is given (still in some specified inertial frame, not a rotating frame) by

$$L = \tfrac{1}{2}m\dot{z}^2 + \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\phi}^2 - U(r, \phi - \Omega t, z). \tag{99}$$

for some fixed angular frequency Ω. Generically, the system is not separately invariant under time translations, nor any spatial translations, nor any spatial rotations. However, systems subject to such uniformly-rotating forces do possess dynamical invariants known as *Jacobi Integrals*, as can be revealed using Noether's Theorem.

Note that under a suitable simultaneous transformation of the azimuthal angle and time (leaving r and z unchanged), namely

$$\phi \to \phi + \epsilon\,\tau\,\Omega, \tag{100a}$$

$$t \to t + \epsilon\,\tau, \tag{100b}$$

for a fixed constant τ and any ϵ, the Lagrangian is completely invariant. The corresponding conservation law is

$$\tfrac{d}{dt}\left[\tau\,L + \tau(\Omega - \dot{\phi})mr^2\dot{\phi} - \tau m\dot{z}^2 - \tau m\dot{r}^2\right] = 0, \tag{101}$$

which after dividing out by a constant $(-\tau)$ and some other simplifications, becomes

$$\tfrac{d}{dt}\left[H - \Omega\,J_z\right] = 0, \tag{102}$$

where $H = \frac{1}{2}m\dot{z}^2 + \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 + U(r, \phi - \Omega t, z)$ is the particle energy in the rotating potential, and $J_z = mr^2\dot{\phi}$ is the component of angular momentum in the direction of the rotating potential.

Although neither the energy nor any component of angular momentum are separately conserved, a certain linear combination of these quantities is conserved. As a check, you are invited to verify this conservation law directly from the equations-of-motion.

3. A Particle in a Traveling-Wave Potential

Suppose instead the Lagrangian for a point particle is given by

$$L = \tfrac{1}{2}m|\dot{\boldsymbol{x}}|^2 - U(\boldsymbol{x} - \boldsymbol{V}t) \tag{103}$$

for some fixed velocity $\boldsymbol{V}$. For example, this might describe a particle moving in a force governed by a traveling wave with phase velocity $\boldsymbol{V}$. Although the physical situation is quite different from our previous example, the mathematics is actually similar.

The system no longer enjoys translational symmetry in space or time separately. However, under the one-parameter group of transformations

$$\boldsymbol{x} \to \boldsymbol{x} + \epsilon\,\tau\,\boldsymbol{V}, \tag{104a}$$

$$t \to t + \epsilon\,\tau, \tag{104b}$$

for any constant τ, and any ϵ (including arbitrarily small ones), the Lagrangian is completely invariant, so Noether's Theorem tell us that

$$\tfrac{d}{dt}\left[\boldsymbol{V}\cdot\boldsymbol{p} - H\right] = 0, \tag{105}$$

where as usual $\boldsymbol{p} = \boldsymbol{p}(t) = m\,\dot{\boldsymbol{x}}(t)$ is the particle's kinetic momentum at time t, and $H = H(t) = \frac{1}{2}m|\dot{\boldsymbol{x}}(t)|^2 + U\big(\boldsymbol{x}(t) - \boldsymbol{V}t\big)$ may be interpreted as the total energy of the particle in the traveling wave at time t, even though this energy is explicitly time-dependent. Although neither linear momentum nor energy are separately conserved, a certain linear combination of these observables is conserved. Since H has explicit time-dependence through U, the corresponding invariant observable has explicit time dependence as well in its functional form, although its value is constant.

Since $\boldsymbol{V}$ is constant vector, we can rearrange this time-dependent conservation law to read

$$\tfrac{dH}{dt}. = \boldsymbol{V} \cdot \tfrac{d\boldsymbol{p}}{dt}. \tag{106}$$

If the external potential is, for example, a dispersion-free *wave packet*, still moving everywhere at fixed velocity $\boldsymbol{V}$, but such that $\lim_{t \to \pm\infty} U(\boldsymbol{x} - \boldsymbol{V}t) = 0$ for any finite $\boldsymbol{x}$, then our conservation law can be integrated to yield:

$$K(+\infty) - K(-\infty) = \boldsymbol{V} \cdot \big[\boldsymbol{p}(+\infty) - \boldsymbol{p}(-\infty)\big], \tag{107}$$

a relation between the total kinetic energy given to the particle by the wave-packet as it passes over the particle, and the net momentum delivered to the particle by the wave packet in the direction of travel of the wave.

V. DISCUSSION

We end with a few comments reiterating what the symmetry/conservation law connection buy us in terms of using physical theories and looking for new ones.

A. What Noether's Theorem Does for Us

It is often easier to look for symmetries than to guess at various forms of conservation laws, and easier to confirm a symmetry than verify the conservation law directly. With the symmetry comes the conservation law, and with the conservation law comes in principle, and frequently in practice, a simplification of the dynamical problem. Every conservation law from a continuous symmetry will reduce either the number of differential equations we have to solve, or their order, i.e., the highest time derivative which appears. With a sufficient number of conservation laws, some dynamical problems can be "reduced to quadrature," meaning that the solution can be written in terms of a number of one-dimensional definite integrals that may either be solved analytically or approximated numerically given the initial conditions. In fact, it is not an exaggeration to say that essentially the only nontrivial exact solutions we can find in classical mechanics arise for problems of this type, with "enough" conservation laws.

1. *Example: Single Degree-of-Freedom Conservative Systems*

For example, consider a particle of mass m moving in *a single* spatial dimension, subject to a force derivable from a time-independent potential energy $U(x)$. The possible shapes of the trajectories are completely fixed as the contours of the Hamiltonian function. Initial conditions determine the conserved value of the total mechanical energy E, specifying on which trajectory the system lies, and an overall timing or phase to say where on the trajectory the system starts. The subsequent relation between position and time can be determined in principle from integrals of the form

$$\int dt = \int \frac{dx}{v} = \pm \int \frac{dx}{\sqrt{2\big(E - U(x)\big)/m}}, \tag{108}$$

with appropriate changes in sign at any *turning-points* where the velocity vanishes.

2. Noether's Theorem in Reverse

Beyond the textbook problems, Noether's Theorem is often invoked in the reverse, or converse direction — guiding our search for possible Lagrangians describing fundamental particles or other systems, that possess certain conservation laws. In order to possess a certain conservation law, the Lagrangian must have the corresponding symmetry, which, together with the rules for how Lagrangians transform under changes of coordinates or reference frames, can often significantly limit the types of terms that can appear in the Lagrangian, and guide physicists to possible new theories which remain consistent with conservation of energy, momentum, electric charge, or other fundamental constraints.

B. Generalizations

In this course and beyond — in classical mechanics, electrodynamics, fluid dynamics, quantum mechanics, etc. — you will make frequent use, explicitly or implicitly, of the connections between symmetries and dynamical invariants. These connections run deeper even than the version of Noether's Theorem presented here.

1. Countable versus Uncountable Degrees-of-Freedom

We have focused on systems which consists of a finite number of particles, or otherwise on at most a countable number of degrees-of-freedom. Noether's theorem can be usefully extended to systems described formally by even an uncountably infinite number of degrees-of-freedom, for example in fluid dynamics or continuum mechanics, where the mechanical systems is described by assigning quantities such as velocity, pressure, etc., to every point in space at a given time, or electrodynamics, where we ascribe values to the electric and magnetic fields at every point in space (and time).

In fact, in such continuum field theories, Noether's Theorem is, if anything, more useful than in discrete particle mechanics. Several non-obvious conservation laws in fluid mechanics, of things called *enstrophy* and *vorticity* in fluid mechanics, arise from Noether symmetries, as does the conservation of *charge* in electromagnetism.

2. Classical versus Quantum Systems

This being a course of classical mechanics, we naturally have focused on, well, classical mechanical systems. But there are versions as well of Noether's Theorem for quantum mechanical systems, and the idea of conservation laws arising from symmetry plays an equally important role in quantum theory. In ordinary (non-relativistic quantum mechanics) symmetries of the Hamiltonian, or of the set of possible wavefunctions lead to conservation laws, whereas in relativistic quantum mechanics we often look to symmetries of the Lagrangian under transformations of the quantum fields, in much the same way as is done for classical field theories.

3. Continuous versus Discrete Systems

Furthermore, there also exist direct connections between *discrete* symmetries, and conservation laws, although in such cases the conserved quantity is most naturally expressed in a manner which is not additive over independent sub-systems (like momentum), but rather multiplicative.

These discrete symmetries and associated conservation laws and related "selection rules" play an especially important role in quantum mechanics. A deep result in quantum theory seems to tells us that if a Lagrangian is fully consistent with special relativity, then it must be be invariant with respect to the combined action of parity, charge conjugation, and time reversal.

4. Conservations Laws Without Geometric Symmetries?

However, there are some conservation laws that are not directly related to Lagrangian symmetries of the essentially *geometric* flavor considered here, but rather to *topological* properties of the space of solutions or boundary conditions, The invariance in time of these so-called *topological charges* is still related to another invariance, and is still analyzed using the machinery of group theory, but the relevant group characterizes topological properties, such as the number of holes, that remain invariant under all continuous deformations, rather than geometric properties like translations or rotations by definite amounts.

5. Higher-Dimensional Spaces

Finally, recall that the symmetries of the action considered here were under transformations of the system's configurations space, and/or of the evolution time. The transformations were velocity-independent, and generalized velocities could not be independently transformed, but inherited their behavior from that of the generalized positions and time. Not all symmetry/conservation law pairs are of this form.

Going back to our very first example: consider a point particle of mass m subject to an attractive inverse-square force, corresponding to a potential energy of the form $U(\boldsymbol{x}) = -\kappa\,|\boldsymbol{x}|^{-1}$. The total energy $E = K + U$ is conserved, as is the total angular momentum $\boldsymbol{J} = \boldsymbol{x} \times \boldsymbol{p}$ about the origin. But another vector-valued quantity is also conserved, the so-called *Laplace-Runge-Lenz* vector:

$$\boldsymbol{A} = \boldsymbol{p} \times \boldsymbol{J} - m\,\kappa\,\hat{\boldsymbol{x}}, \tag{109}$$

which is *nonlinear* in the momenta. This really only adds one more *independent* conserved quantity, not three, since $\boldsymbol{A}$, $\boldsymbol{J}$, and E are related by two additional equations, $\boldsymbol{A}\cdot\boldsymbol{J} = 0$ and $|\boldsymbol{A}|^2 = m^2\kappa^2 + 2mE|\boldsymbol{J}|^2$, but this additional conservation law cannot be derived from the version of Noether's Theorem presented here. Indeed this conservation law is connected to a symmetry, but in a higher-dimensional space, involving both the position and velocity of the particle.

How to handle velocity-dependent transformations and symmetries within the framework of Lagrangian mechanics actually remains a matter of some disagreement However, such symmetries and their associated conservation laws can be naturally formulated from a Hamiltonian perspective in the setting of *phase space*, the space of the generalized positions *and* their conjugate momenta. You will learn much more about this later in the course.

APPENDIX A: LAGRANGIAN GAUGE FREEDOM AND THE EULER-LAGRANGE EQUATIONS

We claim that if $\boldsymbol{q}(t)$ satisfies the Euler-Lagrange equations for some Lagrangian $L(\boldsymbol{q},\dot{\boldsymbol{q}},t)$ and $F(\boldsymbol{q},t)$ is any differentiable function of generalized position and time (but not generalized velocities), then the same trajectory $\boldsymbol{q}(t)$ also satisfies the Euler-Lagrange Equations with respect to the modified Lagrangian $L(\boldsymbol{q},\dot{\boldsymbol{q}},t)+\frac{d}{dt}F(\boldsymbol{q},\dot{\boldsymbol{q}},t)$. Using $L+\frac{d}{dt}F$ in the Euler-Lagrange equations, we have

$$\begin{aligned}\tfrac{d}{dt}\tfrac{\partial}{\partial\dot{\boldsymbol{q}}}\big[L+\dot{F}\big]-\tfrac{\partial}{\partial\boldsymbol{q}}\big[L+F\big]&=\big(\tfrac{d}{dt}\tfrac{\partial L}{\partial\dot{\boldsymbol{q}}}-\tfrac{\partial L}{\partial\boldsymbol{q}}\big)+\big(\tfrac{d}{dt}\tfrac{\partial\dot{F}}{\partial\dot{\boldsymbol{q}}}-\tfrac{\partial\dot{F}}{\partial\boldsymbol{q}}\big)\\&=0+\big(\tfrac{d}{dt}\tfrac{\partial}{\partial\dot{\boldsymbol{q}}}-\tfrac{\partial}{\partial\boldsymbol{q}}\big)\tfrac{d}{dt}F(\boldsymbol{q}(t),t),\end{aligned}\tag{A1}$$

since we are assuming the Euler-Lagrange equations associated with $L(\boldsymbol{q},\dot{\boldsymbol{q}},t)$ itself are satisfied. Since F does not depend explicitly on $\dot{\boldsymbol{q}}(t)$,when acting on F the total time derivative can be written as

$$\tfrac{d}{dt}F=\big(\tfrac{\partial}{\partial t}+\dot{\boldsymbol{q}}\cdot\tfrac{\partial}{\partial\boldsymbol{q}}+\ddot{\boldsymbol{q}}\cdot\tfrac{\partial}{\partial\dot{\boldsymbol{q}}}\big)F=\big(\tfrac{\partial}{\partial t}+\dot{\boldsymbol{q}}\cdot\tfrac{\partial}{\partial\boldsymbol{q}}\big)F.\tag{A2}$$

Applying $\frac{\partial}{\partial\boldsymbol{q}}$ and permuting partial derivatives where possible, we have

$$\tfrac{\partial}{\partial\boldsymbol{q}}\tfrac{d}{dt}F=\big(\tfrac{\partial}{\partial t}\tfrac{\partial}{\partial\boldsymbol{q}}+\dot{\boldsymbol{q}}\cdot\tfrac{\partial}{\partial\boldsymbol{q}}\tfrac{\partial}{\partial\boldsymbol{q}}\big)F=\big(\tfrac{\partial}{\partial t}+\dot{\boldsymbol{q}}\cdot\tfrac{\partial}{\partial\boldsymbol{q}}\big)\tfrac{\partial}{\partial\boldsymbol{q}}F,\tag{A3}$$

while applying $\frac{\partial}{\partial\dot{\boldsymbol{q}}}$ yields

$$\tfrac{\partial}{\partial\dot{\boldsymbol{q}}}\tfrac{d}{dt}F=\tfrac{\partial}{\partial\boldsymbol{q}}F,\tag{A4}$$

which is again independent of $\dot{\boldsymbol{q}}$, so that

$$\tfrac{d}{dt}\tfrac{\partial}{\partial\dot{\boldsymbol{q}}}\tfrac{d}{dt}F=\big(\tfrac{\partial}{\partial t}+\dot{\boldsymbol{q}}\cdot\tfrac{\partial}{\partial\boldsymbol{q}}\big)\tfrac{\partial}{\partial\boldsymbol{q}}F=\tfrac{\partial}{\partial\boldsymbol{q}}\tfrac{d}{dt}F.\tag{A5}$$

Hence

$$\big(\tfrac{d}{dt}\tfrac{\partial}{\partial\dot{\boldsymbol{q}}}-\tfrac{\partial}{\partial\boldsymbol{q}}\big)\tfrac{d}{dt}F(\boldsymbol{q}(t),t)=0,\tag{A6}$$

and we conclude that the Euler-Lagrange equations are indeed invariant under a LAgrangian gauge transformation.

APPENDIX B: EXAMPLE OF INEQUIVALENT LAGRANGIANS GENERATING THE SAME EQUATIONS-OF-MOTION

Consider the usual "kinetic-minus-potential" Lagrangian for two uncoupled, one-dimensional harmonic oscillators of equal mass and spring constants:

$$L=\tfrac{1}{2}m\dot{x}_1^2+\tfrac{1}{2}m\dot{x}_2^2-\tfrac{1}{2}kx_1^2-\tfrac{1}{2}kx_2^2.\tag{B1}$$

Variation with respect to the first coordinate yields the Euler-Lagrange equation:

$$\tfrac{d}{dt}\tfrac{\partial L}{\partial\dot{x}_1}-\tfrac{\partial L}{\partial x_1}=m\ddot{x}_1+kx_1=0,\tag{B2}$$

and similarly variation with respect to the second coordinate generates:

$$\tfrac{d}{dt}\tfrac{\partial L}{\partial\dot{x}_2}-\tfrac{\partial L}{\partial x_2}=m\ddot{x}_2+kx_2=0.\tag{B3}$$

But now consider the Lagrangian

$$\tilde{L}=m\dot{x}_1\dot{x}_2-kx_1x_2.\tag{B4}$$

This in fact generates the equations-of-motion

$$\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{x}_1} - \frac{\partial \tilde{L}}{\partial x_1} = m\ddot{x}_2 + kx_2 = 0, \tag{B5}$$

and:

$$\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{x}_2} - \frac{\partial \tilde{L}}{\partial x_2} = m\ddot{x}_1 + kx_1 = 0. \tag{B6}$$

Somewhat surprisingly, variations with respect to one coordinate generate the equation-of-motion governing the *other* oscillator, but nevertheless the same set of Euler-Lagrange equations is generated by both Lagrangians. Yet these two Lagrangians differ by

$$L - \tilde{L} = \tfrac{1}{2}m\big(\dot{x}_1 - \dot{x}_2\big)^2 - \tfrac{1}{2}k\big(x_1 - x_2\big)^2, \tag{B7}$$

which cannot be expressed as a total time derivative.

APPENDIX C: LEIBNIZ INTEGRAL RULE

The Leibniz integral Rule Tells us how to differentiate a definite integral with respect to a parameter that may appear in both the integrand and the limits of integration. It was a favorite tool of Feynman. Using the Fundamental Theorem of Calculus and the chain rules for partial derivatives, we can deduce:

$$\frac{d}{d\epsilon}\int\limits_{a(\boldsymbol{u},\epsilon)}^{b(\boldsymbol{u},\epsilon)} dt\, f\big(t,\boldsymbol{u},\epsilon\big) = f\big(b(\boldsymbol{u},\epsilon),\boldsymbol{u},\epsilon\big)\,\frac{\partial}{\partial\epsilon}b(\boldsymbol{u},\epsilon) - f\big(a(\boldsymbol{u},\epsilon),\boldsymbol{u},\epsilon\big)\,\frac{\partial}{\partial\epsilon}a(\boldsymbol{u},\epsilon) + \int\limits_{a(\boldsymbol{u},\epsilon)}^{b(\boldsymbol{u},\epsilon)} dt\, \frac{\partial}{\partial\epsilon} f\big(t,\boldsymbol{u},\epsilon\big), \tag{C1}$$

which includes the effects of variation with respect to the parameter ϵ in both the limits of integration and the integrand. Note carefully that in the boundary terms, the partial derivatives with respect to ϵ are taken at fixed values of any additional "free" independent variables $\boldsymbol{u}$, while in the integral the partial derivative of the integrand with respect to the parameter ϵ is taken at fixed values of $\boldsymbol{u}$ *and* the integration variable t.

This is sometimes referred to as" differentiation under the integral sign," which sounds a bit misleading because it also involves derivatives of the integration limits.