List of Problems to Study for the Final Exam

Problem 1. (Sloshing wine)
Consider a wine glass which has a spherically shaped bowl. Assume that this glass is filled halfway with wine, so that the wine itself takes a hemispherical shape. Suppose that some disruption sets the wine oscillating with a small amplitude. Find the period of the small oscillations of the wine in terms of the geometry of the glass. Estimate a numerical value for this period (check your estimate next time you have a glass of wine!).

Problem 2. (Transients)
Find the initial conditions such that an underdamped harmonic oscillator will immediately begin steady-state motion (that is, the attractor motion) under the time-dependent driving $f(t) = f_0 \cos(\omega t)$. Express your answer in terms of f_0, ω_0, ω and β. (You can imagine this would be useful in electrical applications where you don't want transients to effect the behavior of a circuit.)

Problem 3. (Sprung swinging bar)
A uniform bar of mass M and length $2l$ is suspended from one end by a spring of force constant k. The bar can swing freely only in one vertical plane, and the spring is constrained to move only in the vertical direction. Setup the equations of motion in the Lagrangian formulation.

Problem 4. (Minimal surfaces: a special case)
This semester you have found number of geodesics, that is, curves of shortest length given some geometrical constraint. A minimal surface extends this idea to surfaces rather than curves; a minimal surface is a surface that has the smallest possible area given some geometrical constraint. For minimal surfaces the most common constraint is that the boundary of the surface is a given, fixed curve in 3D space. Soap films are minimal surfaces and provide a nice, visualizable context. If you pick up a soap film with a planar circular wire then the soap film takes the shape of the minimal surface, which in this case is just a planar disk. Minimal surfaces are more interesting when you bend your wire into a 3D closed curve and again a soap film will automatically solve the problem for you. In general the minimal surface problem is difficult, so in this problem we will treat a special case.

Construct a surface as follows: Take a curve in the xy-plane, e.g. given by the function $y(x)$, that connects the two points (x_1, y_1) and (x_2, y_2) and rotate it 360° about the y-axis. This is called a surface of revolution and is depicted in Figure 1. The figure suggests a way to find the surface area of such a surface of revolution; the shaded strip has a differential area $dA = 2\pi x ds$ where s is the arc length of the curve and so,

$$A = 2\pi \int_{(x_1,y_1)}^{(x_2,y_2)} x ds.$$

Use this expression to find the curve $y(x)$ which leads to the surface of revolution that is minimal, i.e. has the minimal possible surface area A.[1]

[1] As a pleasant surprise you will find that your answer to this problem is closely related to the answer of another

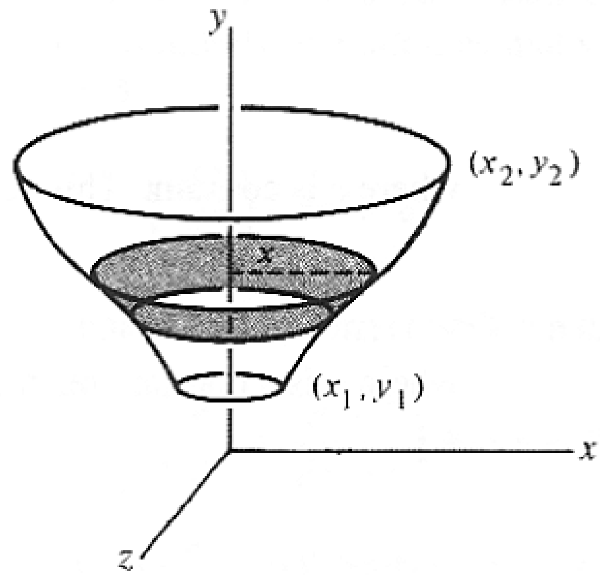


Figure 1: A surface of revolution.

Problem 5. (Rolling and sliding)
A hoop of mass m and radius R rolls without slipping down an inclined wedge of mass M and inclination α, Figure 2. The wedge is free to slide on the plane of the table without friction. The hoop is released from rest at the top of the wedge, which is also initially at rest. If the sloping face of the wedge has a length ℓ how long does it take the hoop to reach the bottom?

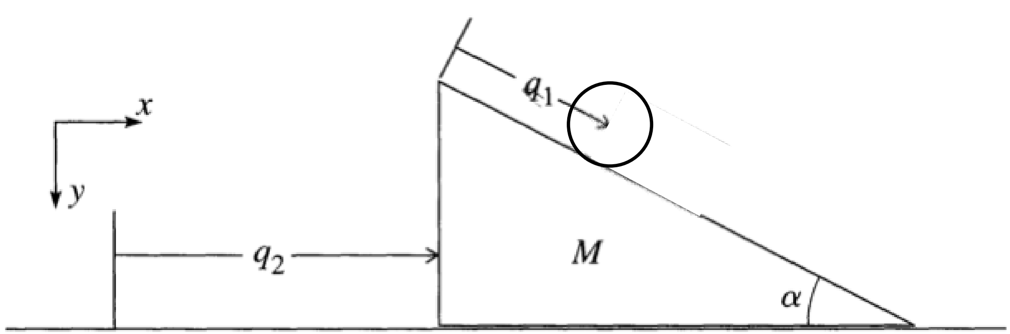
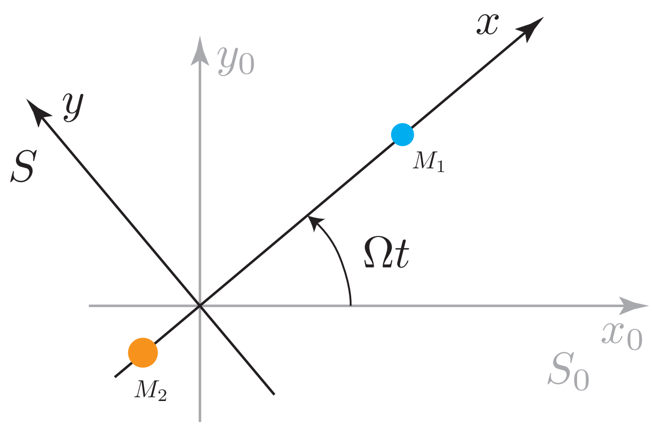


Figure 2: Setup for Problem 5.

calculus of variations problem you have worked on.

Problem 6. (Kepler's law from rotating frames)
Suppose that we have solved the two body problem for the orbit of a planet, of mass M_1, about a star of mass M_2 in the inertial frame S_0 (assume that $M_2 > M_1$ but not necessarily much greater). In terms of the relative coordinate $\mathbf{r}_0 = \mathbf{r}_{10} - \mathbf{r}_{20}$ the orbit is a circle with radius $|\mathbf{r}_0| = R_{\text{sp}}$, the star-planet separation. Take the CM to be the origin of your coordinates, so that $\mathbf{R}_0 = 0$. Also we will work completely in the orbital plane, so you can ignore the z coordinate throughout.

(a) Argue that the planet and the star both move in circles about the CM (you can express $\mathbf{r}_{10}$ and $\mathbf{r}_{20}$ in terms of $\mathbf{r}_0$ to achieve this).
(b) Find the radii of these circles, $|\mathbf{r}_{10}|$ and $|\mathbf{r}_{20}|$, in terms of R_{sp} and the shorthands $\mu_1 = M_1/M$ and $\mu_2 = M_2/M$, where you will recall $M = M_1 + M_2$.
(c) Argue that the period to traverse these two circles is the same for both the planet and the star (you don't need to find a formula for this period yet).



From the arguments that you made in parts **(a)**-**(c)** it is clear that there is a frame, call it S, that rotates around the CM so that both the star and the planet are stationary in this frame (see Figure). That is, the frame S rotates about the CM with the same period as the planet and star orbit it. Assume the frames S_0 and S are initially aligned.

(d) In S the planet still experiences the gravitational attraction of the star. However, the planet is not moving in S. Which (fictitious) force balances the star's gravitational attraction and why is the other one irrelevant?
(e) What direction does the relevant fictitious force point in (your answer should be according to observers in S)?
(f) Use the force balance between the relevant fictitious force and the star's gravitational attraction on the planet to find the frequency of rotation Ω of the frame S in terms of G, M and R_{sp}. This is another way of deriving Kepler's second law for circular orbits.

Problem 7. (Lagrange points L4 and L5)
The setup for this problem is the same as in Problem 6 but now we introduce a small satellite of

mass $m << M_1, M_2$. We assume the satellite is small enough to leave the motion of the star and the planet completely unaffected. The goal of this problem is to find two special points called Lagrange points, in particular we will find the points called L4 and L5. A satellite placed at a Lagrange point of a two body system maintains its relative configuration with the two bodies throughout the orbit. Again it will be useful to do the analysis in the rotating frame S.

The coordinate transformation that connects S to S_0 is the rotation,

$$x_0 = x\cos(\Omega t) - y\sin(\Omega t)$$
$$y_0 = x\sin(\Omega t) + y\cos(\Omega t).$$

Assume the satellite is at a general position $\mathbf{r}_{\text{sat}} = (x, y)$ in S. Write down the Lagrangian and EOMs of the satellite by:
(a) Transforming the kinetic energy from S_0 to the rotating frame using the coordinate transformation given.
(b) Writing down the potential energy of the satellite $U(x, y)$ directly in S (no need to transform for this part). Once you have written this explicitly in terms of the (x, y) coordinates you can simplify it by introducing the shorthands r_{s1} and r_{s2} for the distances from the planet to the satellite and the star to the satellite respectively.
(c) Forming the difference $\mathcal{L} = T - U$ and writing down the two EOM for this Lagrangian.

Now, the condition for our satellite to remain in the same position relative to the other two bodies is that it be fixed in the rotating frame, that is, that it be at an equilibrium in S!
(d) Write down but do not solve the two equations determining the equilibrium positions of the satellite.

There's a nice technique for finding some of the solutions of these two equations that goes as follows:
(e) Find an effective potential $U_{\text{eff}}(x, y)$ whose gradient gives rise to your two equilibrium equations.
(f) Calculate $\mu_1 r_{s1}^2 + \mu_2 r_{s2}^2$ and show that your effective potential can be completely expressed in terms of r_{s1} and r_{s2}, so that $U_{\text{eff}} = U_{\text{eff}}(r_{s1}, r_{s2})$ (Hint: It is useful to show that $\mu_1 + \mu_2 = 1$).
(g) Finally, it turns out that it is enough to find the extrema of $U_{\text{eff}}(r_{s1}, r_{s2})$ to find L4 and L5. Do this and if you were able to solve Problem 1 express your answer in terms of R_{sp}, if not then leave it in terms of G, M and Ω^2. In either case, sketch the two bodies and the two equilibrium positions of the satellite with respect to them in S.

Problem 8. (Rigid body comparisons)
(a) Vector equations are geometrical in nature and, as such, they don't depend on what reference frame they are evaluated in. Given that this is the case, what distinguishes the following two equations for the torque:

$$\mathbf{\Gamma} = \frac{d\mathbf{L}}{dt}$$

and

$$\mathbf{\Gamma} = \dot{\mathbf{L}} + \boldsymbol{\omega} \times \mathbf{L}?$$

A few line answer is sufficient.

(b) A bar of negligible weight and length ℓ has equal mass points m at the two ends. The bar is made to rotate uniformly about an axis passing through the center of the bar and making an angle θ with the bar. From Euler's equations find the components, along the principle axes of the bar, of the torque driving the bar.

(c) From the equation $\mathbf{\Gamma} = d\mathbf{L}/dt$ find the torque along axes fixed in space. Show that these components are consistent with those found in part **(b)**.

Problem 9. (Linear triatomic molecule)
As a model of a linear triatomic molecule (such as CO_2), consider the system shown in Figure 3, with two identical atoms each of mass m connected by two identical springs to a single atom of mass M. You can assume that the system is confined to move in one dimension, along the line of the molecule.

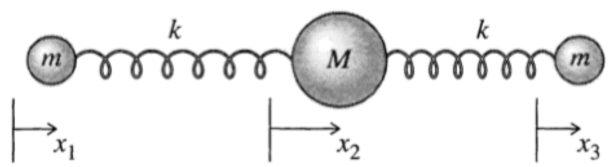


Figure 3: A linear triatomic molecule (such as CO_2).

(a) Write down a Lagrangian and find the normal frequencies of the system.

(b) Find and describe the motion in the normal modes.

Your solution captures a few of the vibrational modes of CO_2, a molecule that is playing an important role in climate change.

(c) If we relaxed the assumption that the motion was one dimensional, what is another motion that would give rise to a normal mode of the CO_2 molecule? (No calculation necessary for this part, I'm just asking you to guess another normal mode. Given more time, you have all the tools necessary to calculate the properties of the mode you've just guessed as well.)

Problem 10. (A charged particle in electric and magnetic fields)
I claim that the Lagrangian,

$$\mathcal{L} = \frac{1}{2}m\dot{\mathbf{r}}^2 - q(V - \dot{\mathbf{r}} \cdot \mathbf{A}),$$

gives rise to the correct motion for a charged particle in electric and magnetic fields. Here q is the charge of the particle, $V(\mathbf{r}, t)$ is the electric potential and $\mathbf{A}(\mathbf{r}, t)$ is the vector potential, i.e.

$\mathbf{E} = -\boldsymbol{\nabla} V - \partial \mathbf{A}/\partial t$ and $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$.

(a) Show that this is a valid Lagrangian by deriving the Lorentz force law,

$$\mathbf{F} = m\ddot{\mathbf{r}} = q(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B}),$$

from it. Derive all three components of this equation at once either by using the vector tools discussed in section or by using the index notation discussed in one of our supplemental notes.

(b) Find the corresponding Hamiltonian. (Again you can use either vector or index methods but in this case it may be fastest to use index methods. This Hamiltonian plays an important role in the quantum mechanics of charged particles.)

(c) Consider a uniform magnetic field $\mathbf{B} = B\hat{z}$, with constant magnitude B (no electric field). Show that the vector potential $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}$ gives rise to this magnetic field. Using this vector potential find all six of Hamilton's equations.