Time Examined

Day 14

Today

I. Ideal Oscillators

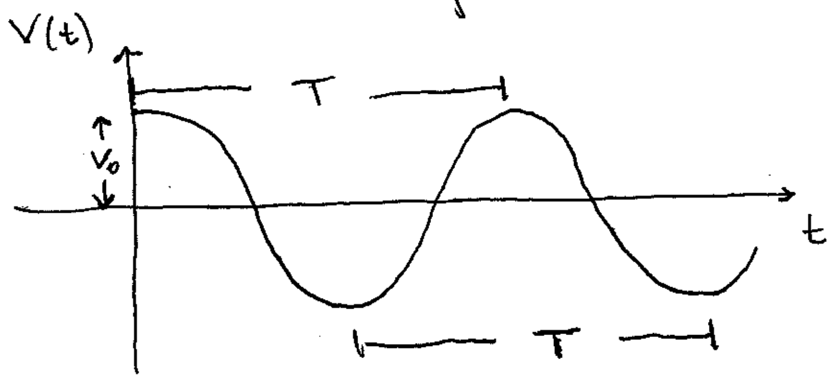
II. Amplitude Noise

III Phase Noise

IV Extracting the frequency of an oscillator

I We have begun to appreciate the connection between clocks and accurate and precise oscillators. It will be useful to give a mathematical characterization of an ideal oscillator. Soon we will relax this idealization and start to see the ways in which clocks can be imperfect.

An ideal oscillator is characterized by its frequency f (or Period T) and its amplitude V_0



Mathematically we write

$$V(t) = V_0 \cos(2\pi f t)$$
$$= V_0 \cos\left(2\pi \cdot \frac{t}{T}\right)$$

where the second equality follows from $f = 1/T$.

Recall that the cosine takes values between 1 and -1 with e.g. $\cos(0) = 1$, $\cos\left(\frac{\pi}{2}\right) = 0$, $\cos(\pi) = -1$,

$\cos\left(\frac{3\pi}{2}\right) = 0$, and $\cos(2\pi) = 1$.

Then, $V(0) = V_0 \cos(0) = V_0$

and $V(T) = V_0 \cos\left(2\pi \frac{T}{T}\right)$

$$= V_0 \cos(2\pi) = V_0$$

The period is the length of time over which the signal $V(t)$ repeats itself. It has units of time and is, for example, measured in seconds. The formula

$$f = \frac{1}{T}$$

gives us two equivalent ways to think about frequency; suppose $T = 0.5s$ then we can say the oscillator goes through 1 cycle in 0.5 s

$$f = \frac{1 \text{ cycle}}{0.5 \text{ s}}$$

or more conveniently we can do out the division and say

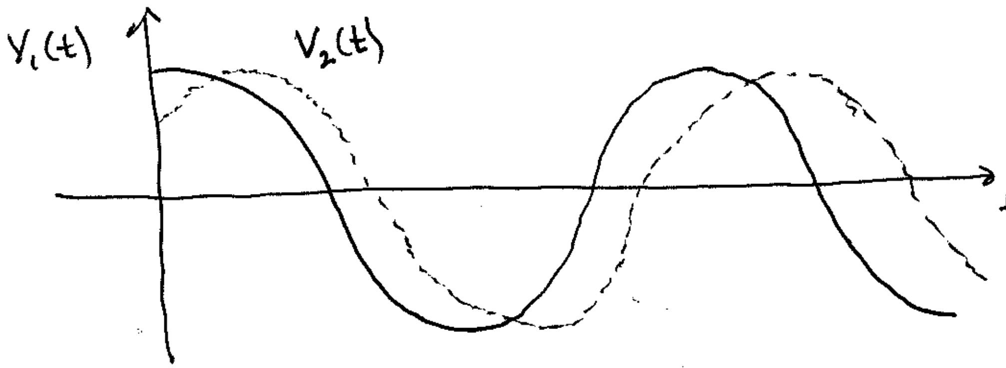
$$f = \frac{1 \text{ cycles}}{0.5 \text{ s}} = \frac{2 \text{ cycles}}{\text{s}}$$

the system goes through 2 cycles per second.

Thinking in terms of cycles helps to under stand the role of the 2π — it is converting a cycle into an angle around the circle, with 2π radians a full circle. Thus in time t the signal sweeps out t/T parts of 2π,

$$V(t) = V_0 \cos\left(2\pi \frac{t}{T}\right).$$

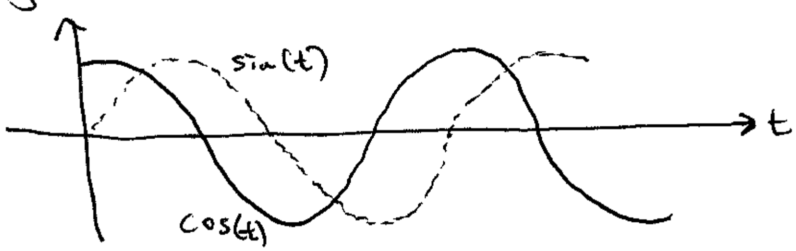
There is one more important freedom to understand for ideal oscillators: suppose that we have two pendula and we start the second one swinging just a bit after the first one, then the two signals will be offset.



2^{nd} pendulum earlier than the first

$$V_3(t) = V_0 \cos(2\pi f t + \delta_0).$$

Wonderfully, the sine function is just a phase shift of a quarter cycle of the cosine function



We call this offset a phase shift and denote it by δ_0.

Mathematically this corresponds to

$$V_1(t) = V_0 \cos(2\pi f t)$$

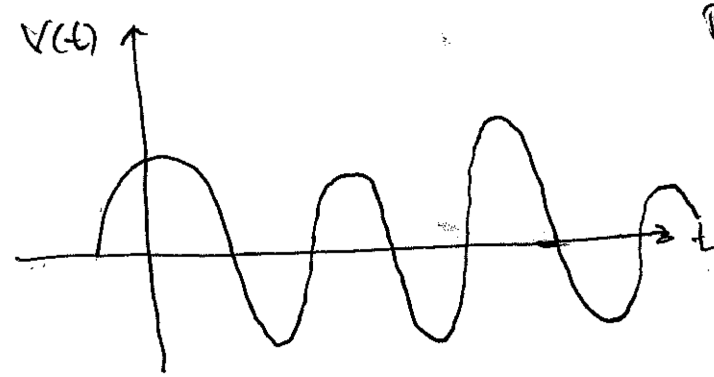
and

$$V_2(t) = V_0 \cos(2\pi f t - \delta_0)$$

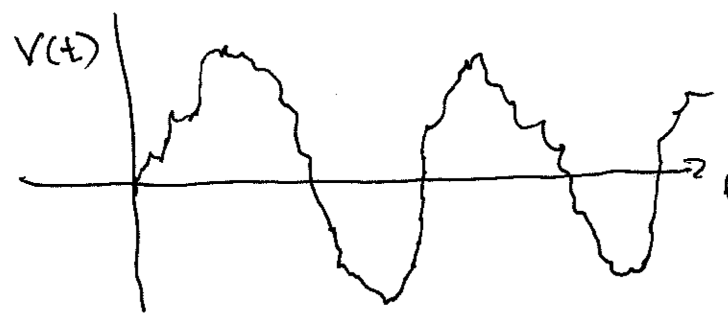
Of course, you could also cause a phase shift in the other direction by releasing the

This completes our discussion of ideal oscillators. Of course, in the laboratory we never encounter ideal oscillators, instead physical oscillators have some kind of noise that disrupts the signal. We'll discuss two types of noise: amplitude noise and phase noise.

II Amplitude Noise: If the length of a pendulum varied in time, e.g. due to changes in temperature, or if there was electronic noise in a circuit generating an oscillating voltage, then the signal can vary in amplitude:



or



Mathematically we capture this by changing to

$$V(t) = (V_0 + \epsilon(t)) \cos(2\pi f t + \delta_0)$$

Interestingly, this kind of noise doesn't matter much for a clock. This is because a counter that is insensitive to these variations is easy to build: you just ensure that it detects the signal a little before the top of the peak. As long as the window that it detects is properly chosen you get the exact same number of counts whether the oscillator is ideal (with no amplitude noise) and in the presence of amplitude noise.

This means we don't have to think about amplitude noise any more.

III Much more important for us are frequency and phase noise. Fortunately these can both be captured together. Suppose our signal is

$$V(t) = V_0 \cos\big(2\pi(\underbrace{f_0 + f(t)}_{\text{frequency noise}})t + \delta_0 + \underbrace{\delta(t)}_{\text{phase noise}}\big)$$

But, notice we can collect all the time dependence inside the cosine into a single term

$$V(t) = V_0 \cos\big(2\pi f_0 t + \delta_0 + \underbrace{[2\pi f(t)t + \delta(t)]}_{\text{call it } \phi(t)}\big)$$

$$= V_0 \cos(2\pi f_0 t + \delta_0 + \phi(t))$$

This means we can model a physical oscillator as an ideal oscillator plus a single phase noise term.

IV Given an oscillator signal

$$V(t) = V_0 \cos(2\pi f_0 t + \delta_0)$$

We can extract f_0 as follows. Call $\Phi(t) = 2\pi f_0 t + \delta_0$. Then

$$\frac{\Delta\Phi}{\Delta t} = \frac{\Phi(t_2) - \Phi(t_1)}{t_2 - t_1}$$

$$= \frac{2\pi f_0 t_2 + \delta_0 - (2\pi f_0 t_1 + \delta_0)}{t_2 - t_1}$$

Cancelling out δ_0 in the numerator and pulling out $2\pi f_0$ gives

$$\frac{\Delta\Phi}{\Delta t} = \frac{2\pi f_0 (t_2 - t_1)}{t_2 - t_1} = 2\pi f_0$$

This shows that we can find f_0 by looking at how Φ changes in time:

$$f_0 = \frac{1}{2\pi}\frac{\Delta\Phi}{\Delta t}.$$

We'll study this in the presence of noise next class.