Time Examined

Day 6

Today

I Last time

II More Consequences of Einstein's two postulates:

(1) Relativity of simultaneity
(2) Time dilation
(3) Length (or Lorentz) contraction
(4) Einstein Velocity addition

In this formula Δt is the time interval between two events, say, D and E,

$$\Delta t = t_D - t_E,$$

as measured in the ground or lab frame. Similarly $\Delta t'$ is the time interval between the same two events, but measured in the moving or train frame

$$\Delta t' = t'_D - t'_E.$$

I We derived the time dilation result

$$\Delta t = \gamma \Delta t'.$$

We observed that

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

is always greater than 1 (and $\gamma = 1$ only when $v = 0$).

Our result shows that moving clocks run slow by a factor of γ.

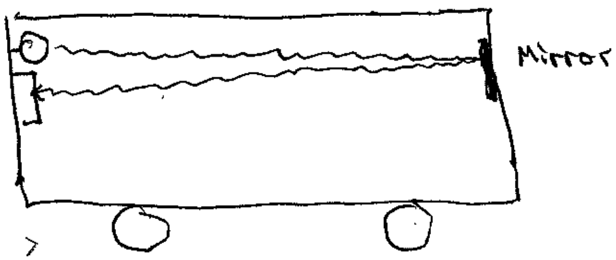
We also stated that

$$L = \frac{1}{\gamma} L'.$$

Today we will derive this result and go more deeply into what it means.

(3) <u>Lorentz contraction</u>

Consider the following train setup

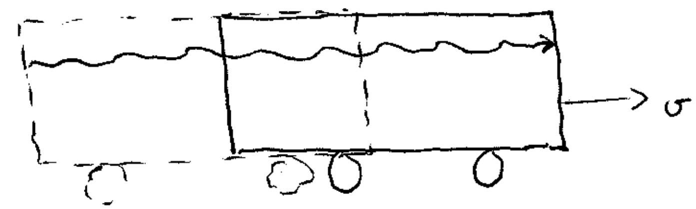


(A) On the train the car is measured to have a length L'. How long does the round trip take?

Well,

$$\Delta t' = \frac{\text{distance}}{\text{speed}} = \frac{2L'}{c}.$$

(B) How long on the ground?



Let Δt_1 = time it takes light to reach mirror at front of car.

Then

length measured on ground

$$\Delta t_1 = \frac{L + v\Delta t_1}{c}$$

Now, we solve for Δt_1, first mulitly through by c

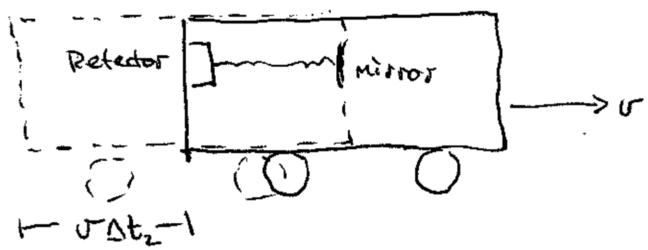
$$c\Delta t_1 = L + v\Delta t_1,$$

but then

$$c\Delta t_1 - v\Delta t_1 = (c-v)\Delta t_1 = L$$

or $\Delta t_1 = \frac{L}{c-v} = \frac{L/c}{1-v/c}$

On the return trip



the train travels $v\Delta t_2$ in a time Δt_2.

So,

$$\Delta t_2 = \frac{L - v\Delta t_2}{c}$$

and multiplying through by c,

$$c\Delta t_2 = L - v\Delta t_2$$

Solving for Δt_2 this time

$$c\Delta t_2 + v\Delta t_2 = (c+v)\Delta t_2 = L$$

or

$$\Delta t_2 = \frac{L}{(c+v)} = \frac{L/c}{1+v/c}$$

Putting each partial trip together gives

$$\Delta t = \Delta t_1 + \Delta t_2 = \frac{L}{c}\left(\frac{1}{1-\frac{v}{c}} + \frac{1}{1+\frac{v}{c}}\right)$$

$$= \frac{L}{c}\left(\frac{1+\frac{v}{c} + 1-\frac{v}{c}}{(1-\frac{v}{c})(1+\frac{v}{c})}\right) = \frac{2L}{c}\frac{1}{1-\frac{v^2}{c^2}}$$

<u>Conclusion</u>: Since $\gamma > 1$ it must be that $L < L'$, moving objects are shortened by a factor of γ.

Length contraction is closely tied to the relativity of simultaneity. What does it mean to measure a length? We want to measure the positions of the two ends of the object at the <u>same time</u>. But,

We can conclude that

$$\Delta t = \frac{2L}{c}\gamma^2.$$

We just found that

$$\Delta t = \gamma \Delta t'$$

and so

$$\frac{2L}{c}\gamma^2 = \gamma\frac{2L'}{c}$$

or canceling 2's, c's and a γ,

$$\boxed{L = \frac{1}{\gamma}L'}$$

We know different observers are going to disagree on what the same time is — hence it is not so surprising they disagree on the length of the object.

Notice that objects always appear longest in their own rest frame: Take $L' = 1\,m$

v	γ	$L = \frac{1}{\gamma} L'$
0 m/s	1	1 m
1/4 c	1.033	0.968 m
1/2 c	1.155	0.866 m
3/5 c	5/4	0.8 m
0.866c	2	0.5 m
0.99 c	7.09	0.141 m

then how can Einstein's 2nd postulate, that the speed of light is the same in all frames, hold? You're right, it can't with this velocity addition rule.

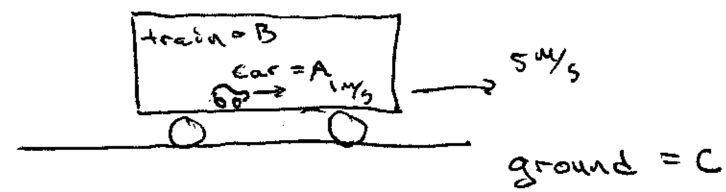
This led Einstein to modify the velocity addition formula:

$$v_{AC} = \frac{v_{AB} + v_{BC}}{1 + \frac{v_{AB} v_{BC}}{c^2}}$$

(4) Einstein Velocity Addition

There is probably something that has bothered you for awhile. If we add velocities like this

$$v_{AC} = v_{AB} + v_{BC}$$



So, a careful answer to HW 1 (3a) would have been

$$v_{car/ground} = \frac{1\,m/s + 5\,m/s}{1 + \frac{1\,m/s \cdot 5\,m/s}{(3\times 10^8\,m/s)^2}}$$

$$= 6\,m/s$$ – a correction in the 17th decimal place.

So, this matters very little for low speeds, but now let's check

What happens for light

$$v_{light/train} = \frac{1\,m/s + 3\times10^8\,m/s}{1 + \frac{1\,m/s \cdot \cancel{3\times10^8\,m/s}}{(3\times10^8\,m/s)^{\cancel{2}}}}$$

$$= \frac{3\times10^8\,m/s}{3\times10^8\,m/s} \cdot \frac{1\,m/s + 3\times10^8\,m/s}{1 + \frac{1\,m/s}{(3\times10^8\,m/s)}}$$

$$= 3\times10^8\,m/s \cdot \underbrace{\frac{1\,m/s + 3\times10^8\,m/s}{(3\times10^8\,m/s) + 1\,m/s}}_{=1}$$

No matter how it is emitted (or detected) the speed of light remains the same!

So,

$$v_{light/train} = 3\times10^8\,m/s = c$$

Do it again in symbols

$$v_{light/ground} = \frac{v_{AB} + c}{1 + \frac{v_{AB}\,\cancel{c}}{c^{\cancel{2}}}}$$

$$= \frac{c}{c} \cdot \frac{v_{AB} + c}{1 + v_{AB}/c}$$

$$= c \cdot \frac{v_{AB} + c}{c + v_{AB}} = c\ !$$