Today's Outline

Lecture 10

Feb 16^{th}, 2012

I Last Lecture

II Geodesics Equation

III Solving the geodesic equation, conservation & symmetry

I Last Lecture

- Usually we calculate with coordinate bases ($\underset{\sim}{e}_\alpha$) and interpret with orthonormal bases ($\underset{\sim}{e}_{\hat{\alpha}}$).

- What relates a^α and $a^{\hat{\beta}}$?

$$a^\alpha = a^{\hat{\beta}} (\underset{\sim}{e}_{\hat{\beta}})^\alpha$$
$$a^{\hat{\gamma}} = a^\delta (\underset{\sim}{e}_\delta)^{\hat{\gamma}}$$

- Because of LIF (Local Inertial Frame) structure the character of ds^2 and τ_{AB} carry over to G.R.

- In practice how will we explore/discover the geometry of spacetime?

Release a bunch of free test particles and allow them to probe the geometry directly by traveling on geodesics

II Geodesics Equation

Variational Principle for Free Test Particle Motion

The world line of a free test particle between two timelike separated points extremizes the proper time between them.

<u>test particle</u>: This is a particle with a small enough mass that its effect on the curvature of spacetime can be neglected.

<u>free particle</u>: free of any influences except gravity.

Trajectories that extremize proper time are called <u>geodesics</u>. In Space these are curves that extremize the length of the curve.

Example: Plane in polar coord.s

$$dS^2 = dr^2 + r^2 d\phi^2$$

parametrize curve by σ, i.e. give $r(\sigma)$ and $\phi(\sigma)$ then

$$S_{AB} = \int_A^B dS = \int_A^B (dr^2 + r^2 d\phi^2)^{1/2}$$

$$= \int_0^1 d\sigma \underbrace{\left[\left(\frac{dr}{d\sigma}\right)^2 + r^2\left(\frac{d\phi}{d\sigma}\right)^2\right]^{1/2}}_{L\left(r,\ \frac{dr}{d\sigma},\ \frac{d\phi}{d\sigma}\right)}$$

Then the E-L equations are,

$$\frac{d}{d\sigma}\left(\frac{\partial L}{\partial\left(\frac{dr}{d\sigma}\right)}\right) = \frac{d}{d\sigma}\left(\frac{1}{L}\frac{dr}{d\sigma}\right) \qquad r\text{-eq.}$$

$$= \frac{\partial L}{\partial r} = \frac{r}{L}\left(\frac{d\phi}{d\sigma}\right)^2$$

$$\frac{d}{d\sigma}\left(\frac{1}{L} r^2 \frac{d\phi}{d\sigma}\right) = 0 \qquad \phi\text{-eq.}$$

Note that, $L = \dfrac{dS}{d\sigma}$

This means that if we parametrize by S we can get rid of the messy Ls:

$$\boxed{\begin{aligned} &r\text{-eq}: \quad \frac{d}{dS}\left(\frac{dr}{dS}\right) = \frac{d^2 r}{dS^2} = r\left(\frac{d\phi}{dS}\right)^2 \\ &\phi\text{-eq}: \quad \frac{d}{dS}\left(r^2\frac{d\phi}{dS}\right) = 0. \end{aligned}}$$

In fact, this works in general — you can <u>drop</u> <u>the squareroot</u>

inside the action whenever you parametrize by proper time (or arclength for spacelike geodesics). ↙ See proof in supplementary notes

Note: this doesn't work in the same manner for lightlike geodesics because $d\tau^2 = -ds^2 = 0$! Moral:

Work with

$$L = -g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta \qquad \dot{x}^\alpha \equiv \frac{dx^\alpha}{d\tau}$$

whenever you can.

Geodesics in general

What are the geodesic eqs of motion (EOM)?

Well take

$$L = -g_{\beta\gamma}(x)\dot{x}^\beta \dot{x}^\gamma \qquad \dot{x}^\alpha \equiv \frac{dx^\alpha}{d\tau}$$

then the Euler-Lagrange Eqs are,

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) - \frac{\partial L}{\partial x^\alpha} = 0$$

Now calculate,

$$\frac{d}{d\tau}\left(-g_{\beta\gamma}\,\delta^\beta_\alpha\,\dot{x}^\gamma - g_{\beta\gamma}\,\dot{x}^\beta\,\delta^\gamma_\alpha\right) = \left(-\frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\dot{x}^\beta\dot{x}^\gamma\right) = 0$$

where $\delta^\beta_\alpha = \begin{cases} 1 & \text{if } \alpha=\beta \\ 0 & \text{if } \alpha\neq\beta \end{cases}$ so that

$$\frac{d}{d\tau}\left(\cancel{-}^{+} g_{\alpha\gamma}\dot{x}^\gamma \cancel{-}^{+} g_{\beta\alpha}\dot{x}^\beta\right) \cancel{+}^{-} \frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\dot{x}^\beta\dot{x}^\gamma = 0$$

Then,

$$\frac{\partial g_{\alpha\gamma}}{\partial x^\beta}\dot{x}^\beta\dot{x}^\gamma + \frac{\partial g_{\beta\alpha}}{\partial x^\gamma}\dot{x}^\beta\dot{x}^\gamma + \overbrace{g_{\alpha\gamma}\ddot{x}^\gamma + g_{\beta\alpha}\ddot{x}^\beta}^{=2g_{\alpha\gamma}\ddot{x}^\gamma} - \frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\dot{x}^\beta\dot{x}^\gamma = 0$$

Collect all the $\dot{x}\dot{x}$ terms together, and divide by 2,

$$g_{\alpha\gamma}\ddot{x}^\gamma + \frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial x^\beta} + \frac{\partial g_{\beta\alpha}}{\partial x^\gamma} - \frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\right)\dot{x}^\beta\dot{x}^\gamma = 0$$

Introduce $g^{\delta\alpha}$ the inverse of $g_{\alpha\gamma}$, i.e. $g^{\delta\alpha}g_{\alpha\gamma} = \delta^\delta_\gamma$

then

$$\ddot{x}^\delta + \underbrace{\frac{1}{2}g^{\delta\alpha}\left(\frac{\partial g_{\alpha\gamma}}{\partial x^\beta} + \frac{\partial g_{\beta\alpha}}{\partial x^\gamma} - \frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\right)}_{\text{call this } \Gamma^\delta_{\beta\gamma}}\dot{x}^\beta\dot{x}^\gamma = 0$$

"Christoffel Symbol"

or

$$\ddot{x}^\delta + \Gamma^\delta_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = 0$$

Recall $\dot{x}^\alpha = \frac{dx^\alpha}{d\tau} = u^\alpha$ so it's also nice to write

$$\ddot{x}^\delta = \frac{du^\delta}{d\tau} = -\Gamma^\delta_{\beta\gamma}u^\beta u^\gamma$$ Geodesic Eq for Timelike Geodesics

Return to our example of the plane in polar coord.s: we can read off the components of the christoffel symbol from

so,

$$\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r}$$

Let's do it again using

$$\Gamma^\delta_{\beta\gamma} = \frac{1}{2}g^{\delta\alpha}\left(\frac{\partial g_{\alpha\gamma}}{\partial x^\beta} + \frac{\partial g_{\beta\alpha}}{\partial x^\gamma} - \frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\right)$$

We have

$$g_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \qquad (A, B = 1, 2)$$

the E.O.M. that we found,

$$\Gamma^r_{\phi\phi} = -r$$

$$\frac{d}{ds}\left(r^2\frac{d\phi}{ds}\right) = r^2\frac{d^2\phi}{ds^2} + 2r\frac{dr}{ds}\frac{d\phi}{ds} = 0$$

$$\Rightarrow \frac{d^2\phi}{ds^2} = -\frac{2}{r}\left(\frac{dr}{ds}\frac{d\phi}{ds}\right)$$

$$= -\frac{1}{r}\left(\frac{dr}{ds}\frac{d\phi}{ds} + \frac{d\phi}{ds}\frac{dr}{ds}\right)$$

so,

$$g^{BC} = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}$$

$$\Gamma^r_{\phi\phi} = \frac{1}{2}g^{rr}\left(\cancel{\frac{\partial g_{r\phi}}{\partial x^\phi}}^{\,0} + \cancel{\frac{\partial g_{\phi r}}{\partial x^\phi}}^{\,0} - \frac{\partial g_{\phi\phi}}{\partial x^r}\right)$$

$$= \frac{1}{2}\cdot 1\cdot(-2r) = -r \quad \checkmark$$

I'll just do one more,

$$\Gamma^{\phi}_{r\phi} = \frac{1}{2} g^{\phi\phi}\left(\frac{\partial g_{\phi\phi}}{\partial x^r} + \frac{\partial g_{r\phi}}{\partial x^{\phi}}^{\nearrow 0} - \frac{\partial g_{r\phi}}{\partial x^{\phi}}^{\nearrow 0}\right)$$

$$= \frac{1}{2}\frac{1}{r^2} 2r = \frac{1}{r} \checkmark$$

Actually because $\Gamma^{\delta}_{\beta\gamma}$ is symmetric in $\beta\gamma$ we have,

$$\Gamma^{\phi}_{r\phi} = \Gamma^{\phi}_{\phi r} = \frac{1}{r}$$

for free.

According to Noether's theorem conservation laws are in 1-to-1 correspondence with symmetries. For example, the conservation of momentum follows from a translational symmetry,

$$L = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m\dot{y}^2 \cancel{+} V(y)$$

no x dependence which means translations in x preserve L. and hence the action

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} = 0$$

$$\Rightarrow \frac{d}{dt}(m\dot{x}) = \frac{d}{dt}(p_x) = 0 \Rightarrow p_x \text{ is conserved}$$

III Solving GE, conservation & symmetry

The geodesic equation (GE) is a set of four coupled, second order, ordinary differential eqns (ODEs). The most general cases need to be solved numerically but conservation laws can simplify the situation.

In G.R. an effective way to capture symmetries is by using a vector. For example, ~~that~~ the flat space metric is invariant under the translation $x^1 \to x^1 + \text{const.}$ and this is captured by the vector

$$\xi^{\alpha} = (0, 1, 0, 0)$$

which points in the direction of this translation.