Today's Outline

Lecture 11
Feb 21st, 2012

I Solving the geodesic Eq (GE), conservation & symmetry

II Survey

III Riemann Normal Coord.s

Announce: Blue books

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(\frac{\partial g_{\delta\gamma}}{\partial x^{\beta}} + \frac{\partial g_{\beta\delta}}{\partial x^{\gamma}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\delta}} \right)$$

Set of 4 coupled, 2nd order ODEs. Conservation laws simplify the solution of these eqn.s.

Noether's theorem

Conservation laws are in 1-to-1 correspondence with symmetries.

For example, conservation of momentum follows from a translational symmetry,

I Solving the GE, conservation & symmetry

Last lecture we found the GE,

$$\ddot{x}^{\alpha} + \Gamma^{\alpha}_{\beta\gamma} \dot{x}^{\beta} \dot{x}^{\gamma} = 0$$

or

$$\frac{du^{\alpha}}{d\tau} + \Gamma^{\alpha}_{\beta\gamma} u^{\beta} u^{\gamma} = 0$$

with Christoffel symbols,

$$L = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2 - V(y)$$

no x-depend. which means translations in x preserve L and hence the action.

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x} = 0$$

$$\Rightarrow \frac{d}{dt}(m\dot{x}) = \frac{d}{dt}(P_x) = 0 \Rightarrow P_x \text{ is conserved.}$$

In G.R. an effective way to capture symmetries is by using a vector, called the generator of the symmetry.

For example, the flat space metric

is invariant under the translation $x^1 \to x^1 + \text{const.}$ and this is captured by the vector

$$\xi^\alpha = (0, 1, 0, 0)$$

that points in the direction of the translation. In G.R. a vector that generates a symmetry, i.e.

$$\tilde{x}^\alpha = x^\alpha + a\,\xi^\alpha$$

leaves the metric (and action) invariant, is called a Killing vector.

$$\Rightarrow \boxed{-\xi \cdot u = \text{const.}}$$

ξ a Killing vector.

Recall the example from last lecture of geodesics in the plane using polar coord.s. We found ~~the~~ GEs

$$\frac{d^2 r}{ds^2} = r\left(\frac{d\phi}{ds}\right)^2$$

$$\frac{d}{ds}\left(r^2 \frac{d\phi}{ds}\right) = 0$$

How does a Killing vector lead to a conservation law?

Use ~~our~~ example to illustrate, a metric invariant under x^1 trans.

$\Rightarrow$ L doesn't depend on x^1

$$\Rightarrow \frac{d}{d\tau}\left(\frac{\partial L}{\partial(dx^1/d\tau)}\right) = 0$$

$(L = -g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta)$

$$\Rightarrow \frac{d}{d\tau}\left(-g_{\alpha\beta}\,\delta^\alpha_1\,\dot{x}^\beta\right) = 0$$

$$\Rightarrow \frac{d}{d\tau}\left(-g_{1\beta}\,\dot{x}^\beta\right) = \frac{d}{d\tau}\left(-g_{\alpha\beta}\xi^\alpha u^\beta\right) = 0$$

The appropriate analog of $u \cdot u = -1$ is $\vec{u} \cdot \vec{u} = 1$ and so,

$$\left(\frac{ds}{ds}\right)^2 = 1 = \left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\phi}{ds}\right)^2$$

The metric doesn't depend on ϕ so,

$$l \equiv \vec{\xi} \cdot \vec{u} = g_{AB}\,\xi^A u^B = r^2 \frac{d\phi}{ds}$$

Then the 1st conservation law gives,

$$\frac{dr}{ds} = \left(1 - \frac{l^2}{r^2}\right)^{1/2}$$

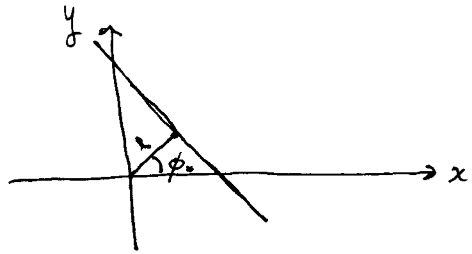
To find the orbit we take,

$$\frac{d\phi/ds}{dr/ds} = \frac{d\phi}{dr} = \frac{\ell}{r^2}\left(1-\frac{\ell^2}{r^2}\right)^{-1/2}$$

$$\Rightarrow \quad \phi = \phi_* + \cos^{-1}\left(\ell/r\right)$$

or

$$r\cos(\phi-\phi_*) = \ell$$



II Survey

Q1: What percentage of the time do you feel you've understood the ~~lectures by their end?~~ main points of a lecture when its ended?

Q2: Is the lecture moving too fast too slow, or at the right pace?

Comments on null geodesics:

For light $ds^2 = 0$

$$\Rightarrow \quad \underline{u}\cdot\underline{u} = g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0$$

λ an affine parameter. Only 1 eq., to derive null geodesics need relativistic E o M but can guess,

$$\frac{d^2x^\alpha}{d\lambda^2} = -\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda}$$

and it's correct!

Q3: What would you change were you teaching the course? What's going well?

Q4: Accelerated observers feel the effects of their acceleration, e.g. braking. Einstein: "Because of this we feel compelled at the present juncture to grant a kind of absolute physical reality to nonuniform motion, in opposition to the general principle of relativity!"

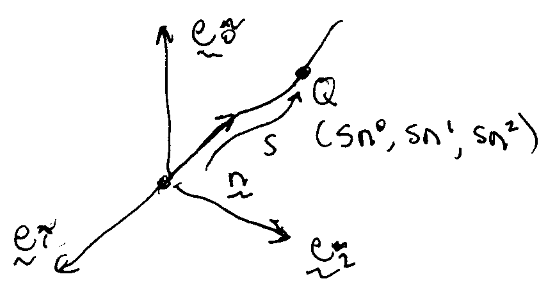
What does Einstein mean by the general principle of relativity? Explain his quote.

~~III Riemann~~ Additional Comments:

III Riemann Normal Coord.s

We now have the tools to construct these coords. Let's do it.

- Pick P.
- Erect a orthonormal frame



- Follow geodesics a distance s or proper time τ.

We have been discussing local inertial frames (LIF)

$$g_{\alpha\beta}(x_p) = \eta_{\alpha\beta} \qquad \left.\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\right|_{x=x_p} = 0$$

Note that were we to find such coordinates

$$\left.\Gamma^\alpha{}_{\beta\gamma}\right|_p = 0$$

$$\Rightarrow \left.\frac{d^2 x^\alpha}{d\tau^2}\right|_p = 0 \Rightarrow \text{straight line motion near P.}$$

- Label points by

$$x^\alpha = s\, n^\alpha$$

Gives unique labels until geodesics cross – restrict to this neighborhood of P. Now,

$$g_{\alpha\beta}(x_p) = \eta_{\alpha\beta} \checkmark$$

✓ coordinate basis = an orthonormal basis

and

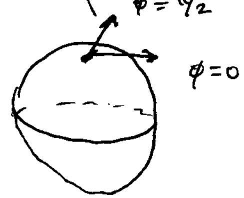
$$\frac{d^2 x^\alpha}{ds^2} = \frac{d}{ds}(n^\alpha) = 0 = \Gamma^\alpha{}_{\beta\gamma} n^\beta n^\gamma$$

but this holds for all n^β, n^γ, so

$$\Gamma^{\gamma}_{\beta\gamma}=0 \quad \Rightarrow \quad \left.\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\right|_{x_p}=0$$

This ~~method~~ coordinate system, which yields an LIF is called the Riemann normal coordinate system.

<u>Example</u>: North pole of sphere again:



$n^A = (\cos\phi, \sin\phi)$

and arc length of great circle is $s = a\theta$, so,

$$x^A = (a\theta\cos\phi,\ a\theta\sin\phi)$$

Look familiar?!