Lecture 12

Feb 28th, 2012

Today's Outline

I Survey Results

II Brief Recap of Geodesics

III Central Force Problem

I Survey Results

see slides

II Brief Recap of Geodesics

- Arrived at Geodesic Eqn (GE) through a variational principle:

$$\frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma = 0$$

Γ are the Christoffel symbols.

- Easier to solve the GE when there are symmetries characterized by ξ a killing vector with corresponding cons. law

$$\xi \cdot u = \text{const.}$$

- Geodesics can be used to construct the coords of a locally inertial frame with

$$g_{\alpha\beta}(x_p) = \eta_{\alpha\beta} \qquad \left.\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\right|_{x=x_p} = 0.$$

Called Riemann normal coord.s.

III Central Force Problem ~~Schwarzschild Geometry~~

In this lecture, and the next two, we begin our exploration of the Schwarzschild geometry:

$$ds^2 = -\left(1-\frac{2GM}{c^2 r}\right)(c\,dt)^2 + \left(1-\frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

This metric describes the geometry of a spherically symmetric source of curvature.

In G.R. mass is a subtle concept; recall our example of a barrel of masses from the first lecture. Note that if $GM/c^2 r$ is small we can expand the ~~first~~ second term to get

$d\Omega^2 \equiv d\theta^2 + \sin^2\theta \, d\phi^2$

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)(c\,dt)^2 + \left(1 + \frac{2GM}{c^2 r}\right)dr^2 + r^2 d\Omega^2$$

Very similar to

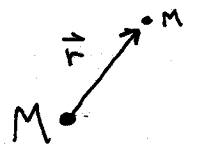
$$ds^2 = -\left(1 + \frac{2\Phi}{c^2}\right)(c\,dt)^2 + \left(1 - \frac{2\Phi}{c^2}\right)(dx^2 + dy^2 + dz^2)$$

and suggests the identifications $\Phi = -\frac{GM}{r}$

and interpretation of M as the mass of the spherically sym. body. These turn out to be correct at lowest order.

As promised we will explore this geometry by examining its geodesics. Before doing this it will be very useful to remind ourselves of how central forces work in non-relativistic physics.

Central Force Problem



$V(r)$ potential energy only depends on $r = |\vec{r}|$.

Because of conservation of ~~angular~~ mom. $\vec{L} = \vec{r} \times \vec{p}$, motion is confined to a plane. So, we'll use polar coords in this plane

(r, ϕ) : $\quad v_r = \dot{r} \quad , \quad v_\phi = r\dot{\phi} \qquad \left(\cdot \equiv \frac{d}{dt} \text{ non-rel.}\right)$

$$\Rightarrow \quad v^2 = v_r^2 + v_\phi^2 = \dot{r}^2 + r^2\dot{\phi}^2$$

Conservation of Energy:

$$E = \text{K.E.} + \text{P.E.} = \tfrac{1}{2} m v^2 + V(r)$$

$$= \tfrac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) + V(r)$$

Conservation of ang. mom.:

$$L = m r^2 \dot{\phi}$$

Eliminate $\dot{\phi}$: $\quad \dot{\phi} = \frac{L}{m r^2}$

$$\Rightarrow \quad E = \tfrac{1}{2} m \dot{r}^2 + \tfrac{1}{2} m r^2 \frac{L^2}{m^2 r^4} + V(r)$$

$$\boxed{E = \tfrac{1}{2} m \dot{r}^2 + \frac{L^2}{2 m r^2} + V(r)}$$

We only want the shape of the orbit; use ϕ (instead of t) as the indep. variable:

$$\dot r = \frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{L}{mr^2}\frac{dr}{d\phi}$$

Change to $u \equiv 1/r$ $\left(r = \frac{1}{u}\right) \rightarrow u(\phi)$

$$\frac{dr}{d\phi} = \frac{dr}{du}\frac{du}{d\phi} = -\frac{1}{u^2}\frac{du}{d\phi}$$

$$\dot r = \frac{L}{m}u^2\left(-\frac{1}{u^2}\frac{du}{d\phi}\right) = -\frac{L}{m}\frac{du}{d\phi}$$

+ : coming in for increasing ϕ

− : going out for increasing ϕ

The Kepler Problem:

$$V(r) = -\frac{GMm}{r} = -GMmu$$

$$\frac{du}{d\phi} = \sqrt{\frac{2mE}{L^2} + \frac{2GMm^2}{L^2}u - u^2} = \sqrt{I(u)}$$

$$I(u) \equiv \frac{2mE}{L^2} + \frac{2GMm^2}{L^2}u - u^2$$

Intro notation $u' = \frac{du}{d\phi}$,

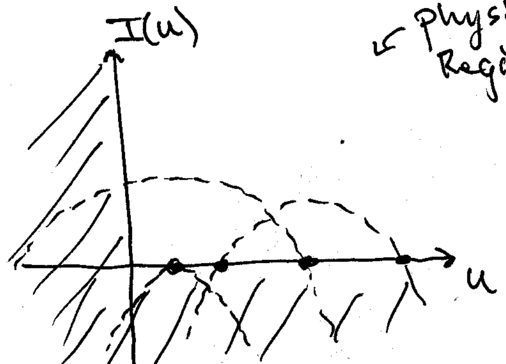
$$E = \frac{1}{2}m\frac{L^2}{m^2}u'^2 + \frac{L^2u^2}{2m} + V$$

$$= \frac{L^2}{2m}\left(u'^2 + u^2\right) + V$$

$$\Rightarrow \boxed{u'^2 = \frac{2m}{L^2}(E-V) - u^2}$$

or

$$u' = \pm\sqrt{\frac{2m}{L^2}(E-V) - u^2}$$



physical Region

$u = \frac{1}{r} > 0$ (physically)

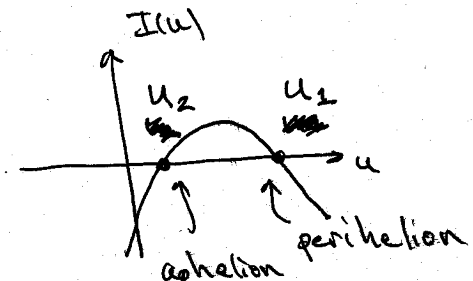
$I(u) \geq 0$, because

$$\frac{du}{d\phi} = \sqrt{I}$$

Two Cases:

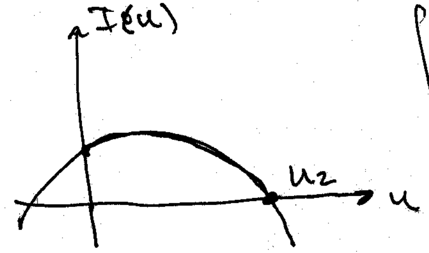
$u_2 > 0$

orbits bounded

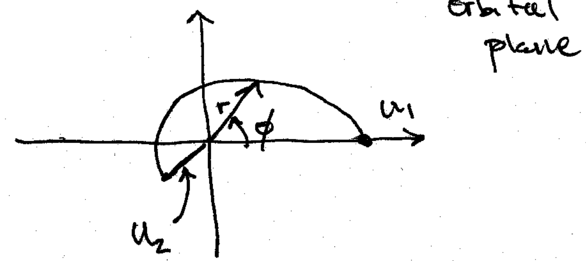


$u_1 < 0$

unbounded - (scattering)



Edit
↓
change all
$u_2 \to u_1$
$u_1 \to u_2$



<u>Case 1</u> (orbits) $u_1 > 0$

$$I(u) = (u - u_1)(u_2 - u)$$

$$\Rightarrow \frac{du}{d\phi} = \sqrt{(u-u_1)(u_2-u)}$$

$$\Phi = \int d\phi = \int_{u_1}^{u_2} \frac{du}{\sqrt{(u-u_1)(u_2-u)}}$$

From table,

$$\int \frac{du}{\sqrt{(u-u_1)(u_2-u)}} = \sin^{-1}\left(\frac{2u - u_2 - u_1}{u_2 - u_1}\right)$$

$$\Rightarrow \Phi = \sin^{-1}\left(\frac{u_2 - u_1}{u_2 - u_1}\right) - \sin^{-1}\left(\frac{u_1 - u_2}{u_2 - u_1}\right)$$

$$= \sin^{-1}(1) - \sin^{-1}(-1) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

Orbit is an ellipse with origin at one focus