Lecture 13

~~Feb~~ Mar 1st, 2012

Today's Outline

I Wrap up Central Force Problem

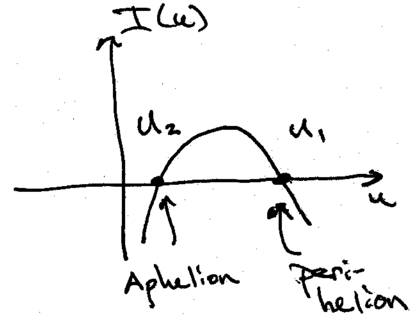
II Schwarzschild: Energy & Effective potential

III Precession

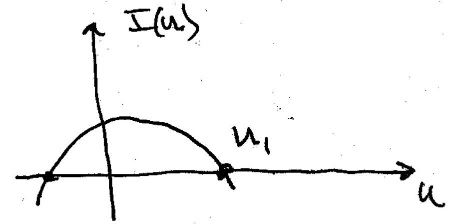
I Wrap up Central Force Problem

Two Cases:

$u_2 > 0$
orbits bounded

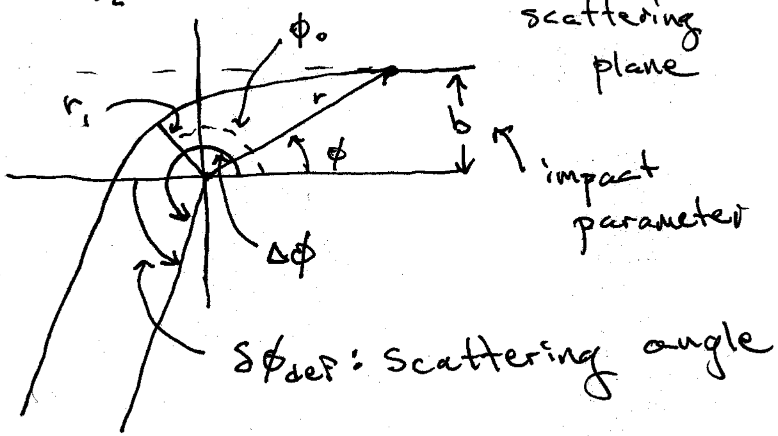


$u_2 < 0$
Unbounded
(scattering)



We did case 1, let's do case 2:

$u_2 < 0$



$$\frac{du}{d\phi} = \sqrt{I(u)} = \sqrt{(u-u_2)(u_1-u)}$$

$$\Rightarrow \Delta\phi = 2\phi_0 = 2\int_0^{u_1} \frac{du}{\sqrt{(u-u_2)(u_1-u)}}$$

$$= 2\left[\sin^{-1}\left(\frac{u_1-u_2}{u_1-u_2}\right) - \sin^{-1}\left(\frac{-u_1-u_2}{u_1-u_2}\right)\right]$$

$$= 2\left[\sin^{-1}(1) + \sin^{-1}\left(\frac{u_1+u_2}{u_1-u_2}\right)\right]$$

$$= \pi + 2\sin^{-1}\left(\frac{u_1+u_2}{u_1-u_2}\right)$$

But $\delta\phi_{def} = \Delta\phi - \pi$,

$$\Rightarrow \quad \delta\phi_{def} = 2\sin^{-1}\left(\frac{u_1+u_2}{u_1-u_2}\right)$$

or

$$\boxed{\sin\left(\frac{\delta\phi_{def}}{2}\right) = \frac{u_1+u_2}{u_1-u_2}}$$

Plug in $E|_\infty = \frac{1}{2}mv^2$ and $L = mvb$ to find u_1, u_2 and classical scattering result

$$\cot\left(\frac{\delta\phi_{def}}{2}\right) = \frac{bv^2}{GM}.$$

II Schwarzschild: Energy & Effective Potential

Sch. spacetime has two important symmetries:

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

It is static, i.e. independent of time, and hence

$\xi^\alpha = (1,0,0,0)$ is a killing vector.

It is also spherically symmetric for constant t and r. One generator is ϕ-translations, i.e.

$\eta^\alpha = (0,0,0,1)$ is a killing vector.

To study geodesics we'll use the conserved quantities arising from these symmetries,

$$e \equiv -\underset{\sim}{\xi}\cdot\underset{\sim}{u} = -g_{\alpha\beta}\xi^\alpha u^\beta$$

$$= -g_{00}\xi^0 u^0 = \left(1-\frac{2M}{r}\right)\frac{dt}{d\tau}$$

conserved Energy per unit test mass

$$l \equiv \underset{\sim}{\eta}\cdot\underset{\sim}{u} = g_{\alpha\beta}\eta^\alpha u^\beta$$

$$= g_{33}\eta^3 u^3 = r^2\sin^2\theta\,\frac{d\phi}{d\tau}$$

Ang. Mom. per unit rest mass

If initially $\phi = \frac{d\phi}{d\tau} = 0$ then ϕ is always zero and motion stays in the "plane" $\phi = 0$.

Reorient so that $\theta = \pi/2$ and $u^\theta = 0$

As always, we use

$$\underset{\sim}{u}\cdot\underset{\sim}{u} = -1$$

$$-1 = g_{\alpha\beta}u^\alpha u^\beta = -\left(1-\frac{2M}{r}\right)(u^t)^2 + \left(1-\frac{2M}{r}\right)^{-1}(u^r)^2 + r^2(u^\phi)^2$$

$$\Rightarrow -\left(1-\frac{2M}{r}\right)^{-1} e^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + \frac{\ell^2}{r^2} = -1$$

$$\Rightarrow -e^2 + \left(\frac{dr}{d\tau}\right)^2 + \frac{\ell^2}{r^2}\left(1-\frac{2M}{r}\right) = -\left(1-\frac{2M}{r}\right)$$

$$\Rightarrow e^2 = \left(\frac{dr}{d\tau}\right)^2 + \left(1-\frac{2M}{r}\right)\left(1+\frac{\ell^2}{r^2}\right)$$

Subtract 1 from each site and mult. by $\frac{1}{2}$,

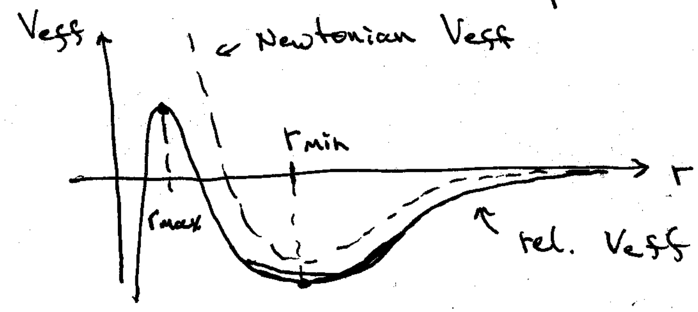
$$\mathcal{E} \equiv \frac{e^2-1}{2} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\left[\left(1-\frac{2M}{r}\right)\left(1+\frac{\ell^3}{r^2}\right)-1\right]$$

This is now of the form

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r)$$

(everything per unit mass) with

$$V_{eff} = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3}$$



Can calculate r_{min} and r_{max},

$$r_{\substack{min\\max}} = \frac{\ell^2}{2M}\left[1 \pm \sqrt{1-12\left(\frac{M}{\ell}\right)^2}\right]$$

Note that for $\ell < \sqrt{12}\,M$ extrema disappear.

<u>Example</u>: Stable Circular Orbits

Circular orbits can occur at r_{min} and r_{max} but are only <u>stable</u> at r_{min}

For given ℓ there's only one r_{min} but by varying ℓ we can ask for the innermost stable circ. orbit (ISCO). The extrema coalesce and dissappear when $\ell = \sqrt{12}\,M$ and hence

$$r_{ISCO} = \frac{12M^2}{2M} = 6M$$

Another calculation: What's $\Omega = \frac{d\phi}{dt}$ for circular orbits?

$$\Omega = \frac{d\phi}{dt} = \frac{d\phi/d\tau}{dt/d\tau} = \frac{1}{r^2}\left(1-\frac{2M}{r}\right)\left(\frac{\ell}{e}\right)$$

Using r_{min} and $e^2 = \left(1-\frac{2M}{r}\right)\left(1+\ell^2/r^2\right)$ some algebra gives,

$$\frac{\ell}{e} = (Mr)^{1/2}\left(1-\frac{2M}{r}\right)^{-1} \quad \text{for circular orbits}$$

Then,

$$\Omega = \frac{(Mr)^{1/2}}{r^2}$$

$$\Rightarrow \boxed{\Omega^2 = \frac{M}{r^3}} \quad \text{circular orbits in Sch. coords.}$$

Four velocity, $u^t\Omega = d\phi/d\tau$

$$u^\alpha = u^t(1, 0, 0, \Omega)$$

$$\underset{\sim}{u}\cdot\underset{\sim}{u} = -1 = g_{00}u^{t2} + g_{33}u^{t2}\Omega^2$$

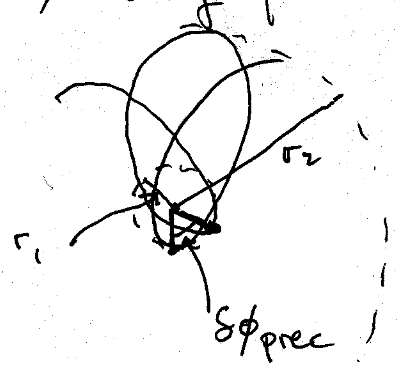
$$= -\left(1-\frac{2M}{r}\right)u^{t2} + r^2\frac{M}{r^3}u^{t2}$$

$$\Rightarrow u^{t2}\left(1-\frac{2M}{r}-\frac{M}{r}\right) = 1$$

$$\Rightarrow \boxed{u^t = \left(1-\frac{3M}{r}\right)^{-1/2}} \quad \text{circular orbits}$$

III Precession

Because of the relativistic correction $-\frac{M\ell^2}{r^3}$ bound orbits no longer close into ellipses! Instead, they precess,



Just as before

$$\frac{dr}{d\tau} = \sqrt{2(\mathcal{E} - V_{eff})}$$

$$\Rightarrow \left|\frac{du}{d\phi}\right| = \frac{1}{\ell}\frac{dr}{d\tau} = \frac{\sqrt{2(\mathcal{E}-V_{eff})}}{\ell}$$

$$\Rightarrow I(u) = \frac{2}{\ell^2}(\mathcal{E}-V_{eff})$$

$$= \frac{2\mathcal{E}}{\ell^2} + \frac{2M}{\ell^2}u - \frac{\cancel{\ell^2}^{\,1}}{\cancel{\ell^2}}u^2 + 2Mu^3$$