Lecture 14
Mar 6th, 2012

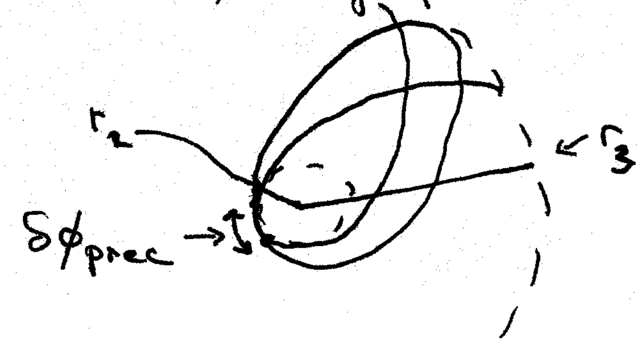
Today's Outline

I Precession of the Perihelion of Mercury

II Deflection of Starlight

I Precession

Because of the relativistic correction $-\frac{M\ell^2}{r^3}$, bound orbits no longer close into ellipses! Instead, they precess,



Just as before

$$\frac{dr}{d\tau} = \sqrt{2(\mathcal{E} - V_{eff})}$$

$$\Rightarrow \left|\frac{du}{d\phi}\right| = \frac{1}{\ell}\frac{dr}{d\tau} = \frac{1}{\ell}\sqrt{2(\mathcal{E} - V_{eff})}$$

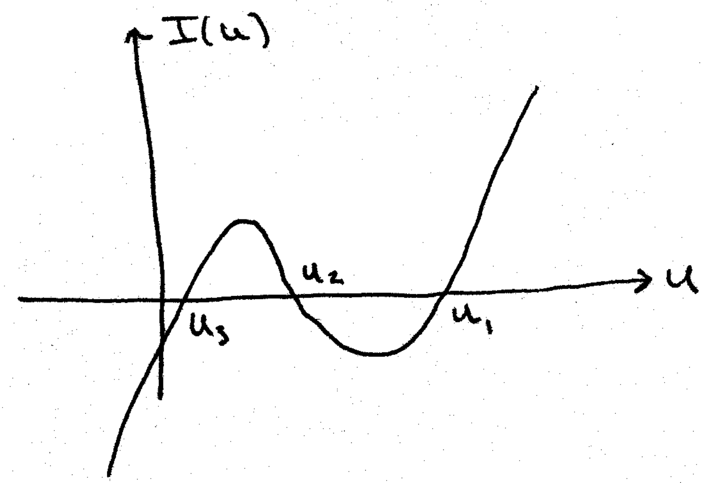
$$\Rightarrow I(u) = \frac{2}{\ell^2}(\mathcal{E} - V_{eff})$$

$$= \frac{2\mathcal{E}}{\ell^2} + \frac{2M}{\ell^2}u - \cancel{\frac{\ell^2}{\ell^2}}^{\,1} u^2 + 2Mu^3$$

and let $a \equiv 2M$, $I(u) = \frac{2\mathcal{E}}{\ell^2} + \frac{a}{\ell^2}u - u^2 + au^3$

Then

$$u' \equiv \frac{du}{d\phi} = \sqrt{I(u)}$$



$$I(u) = 2M(u-u_3)(u_2-u)(u_1-u)$$

$$= a\,(u-u_3)(u_2-u)(u_1-u)$$

Expand,

$$I(u) = a(u-u_3)(u^2 - u_1u - u_2u + u_1u_2)$$
$$= au^3 + (-au_3 - au_1 - au_2)u^2 + \cdots$$
$$= au^3 - a(u_1+u_2+u_3)u^2 + \cdots$$

We must have,

$$u_1+u_2+u_3 = \frac{1}{a} \Rightarrow \boxed{u_1 = \frac{1}{a} - u_2 - u_3}$$

Recall,

so that $r \gg r_1 \Rightarrow u \ll u_1$. Then,

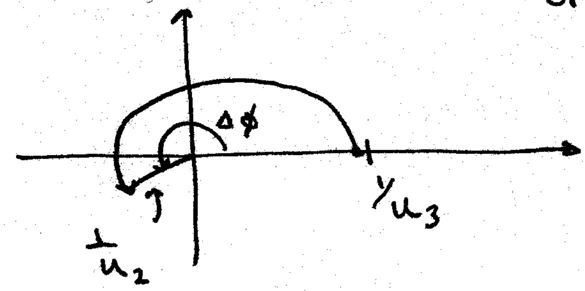
$$(u_1-u)^{-1/2} = u_1^{-1/2}\left(1-\frac{u}{u_1}\right)^{-1/2} \approx u_1^{-1/2}\left(1+\frac{u}{2u_1}\right).$$

$$\Delta\phi = \int_{u_3}^{u_2} \frac{1}{\sqrt{I}}\,du \approx \frac{1}{\sqrt{au_1}}\int_{u_3}^{u_2} \frac{du}{\sqrt{(u-u_3)(u_2-u)}}\left(1+\frac{u}{2u_1}\right)$$

We already encountered the first of these integrals,

$$\int_{u_3}^{u_2} \frac{du}{\sqrt{(u-u_3)(u_2-u)}} = \sin^{-1}\left(\frac{2u-u_3-u_2}{u_2-u_3}\right)\Bigg|_{u_3}^{u_2}$$

orbital plane



So

$$\delta\phi_{prec} = 2(\Delta\phi - \pi).$$

To ease calculation we take the weak field approximation; namely we're far from the source,

$$= \sin^{-1}(1) - \sin^{-1}(-1)$$
$$= \pi$$

Now, to integrate

$$\int_{u_3}^{u_2} \frac{u\,du}{\sqrt{(u-u_3)(u_2-u)}}$$

note that

$$\frac{d}{du}\left(\sqrt{(u-u_3)(u_2-u)}\right) = \frac{1}{2}\frac{1}{\sqrt{(u-u_3)(u_2-u)}} \times (-2u + u_2 + u_3)$$

So,

$$\int_{u_3}^{u_2} \frac{u\,du}{\sqrt{(u-u_3)(u_2-u)}} = -\sqrt{(u-u_3)(u_2-u)}\Big|_{u_3}^{u_2} + \frac{1}{2}(u_2+u_3)\int_{u_3}^{u_2}\frac{du}{\sqrt{(u-u_3)(u_2-u)}}$$

$$= \frac{1}{2}(u_2+u_3)\pi$$

Pulling together,

$$\Delta\phi = \frac{1}{\sqrt{au_1}}\left\{\pi + \frac{1}{2u_1}\cdot\frac{1}{2}\pi(u_2+u_3)\right\}$$

$$= \frac{\pi}{\sqrt{au_1}}\left\{1+\frac{1}{4}\frac{(u_2+u_3)}{u_1}\right\}$$

Then,

$$\delta\phi_{prec} = 2\Delta\phi - 2\pi$$

$$= \pi\frac{3a}{2}(u_2+u_3)$$

$$= \boxed{\frac{3\pi GM}{c^2}\left(\frac{1}{r_2}+\frac{1}{r_3}\right)}$$

If we approximate r_2, r_3 by their Keplerian values, $r_2 \simeq a(1-\epsilon)$, $r_3 \simeq a(1+\epsilon)$ then

semi major axis $\neq 2M$

$$\delta\phi_{prec} \simeq \frac{3\pi GM}{c^2 a}\left(\frac{1}{1-\epsilon}+\frac{1}{1+\epsilon}\right) = \boxed{\frac{6\pi G}{c^2}\frac{M}{a(1-\epsilon^2)}}$$

and putting in

$$a u_1 = \left(\frac{1}{a} - u_2 - u_3\right)a$$

$$\Delta\phi = \frac{\pi}{\sqrt{1-a(u_2+u_3)}}\left\{1+\frac{1}{4}\frac{a(u_2+u_3)}{1-a(u_2+u_3)}\right\}$$

$$\simeq \pi\left(1+\frac{a}{2}(u_2+u_3)\right)\left(1+\frac{a}{4}(u_2+u_3)(1+a(u_2+u_3))\right)$$

1st order in 2nd term

$$\simeq \pi\left(1+\frac{a}{2}(u_2+u_3)\right)\left(1+\frac{a}{4}(u_2+u_3)\right)$$

1st order

$$\simeq \pi\left(1+\frac{3a}{4}(u_2+u_3)\right)$$

Putting in the parameters for Mercury and the sun

$$\delta\phi_{prec} = 42.98''/\text{century}$$

The measured precession is

$$\delta\phi = 5599.74'' \pm 0.41''/\text{century}$$

Subtract out the Newtonian effects, e.g. precession of the equinoxes

$$\delta\phi_{eq} = 5025.64'' \pm .50''/\text{century}$$

and the only unexplained precession

is
$$\delta\phi = 42.98'' \pm 0.04''/\text{century} \quad !$$

II Deflection of Starlight

We proceed in a very similar manner: combine

$$e \equiv -\underset{\sim}{\xi}\cdot\underset{\sim}{u} = \left(1-\frac{2M}{r}\right)\frac{dt}{d\lambda}$$ affine parameter

$$\ell \equiv \underset{\sim}{\eta}\cdot\underset{\sim}{u} = r^2\sin^2\theta\,\frac{d\phi}{d\lambda}$$

and

$$\underset{\sim}{u}\cdot\underset{\sim}{u} = g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0 \qquad \text{(new!)}$$

to find

$$\frac{e^2}{\ell^2} \equiv \frac{1}{b^2} = \frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{eff}(r)$$

with

$$W_{eff} = \frac{1}{r^2} - \frac{2M}{r^3} = \frac{1}{r^2}\left(1-\frac{2M}{r}\right).$$

Light ray geodesics only depend on the ratio

$$b = |\ell/e|$$

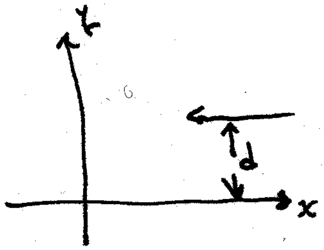
not on ℓ and e separately (due to freedom of choice in affine parameter).

At ∞ can choose cartesian coord.s such that

$$x = r\cos\phi\,, \quad y = r\sin\phi \qquad \left(\theta = \frac{\pi}{2} \text{ "equatorial plane"}\right)$$

then,

$$b \equiv \left|\frac{\ell}{e}\right| \overset{r \gg 2M}{=} \frac{r^2\, d\phi/d\lambda}{dt/d\lambda} = r^2\frac{d\phi}{dt}$$



$$\phi \approx \frac{d}{r}$$

$$\frac{dr}{dt} \approx \frac{dx}{dt} = -c = -1$$

so,

$$b \approx r^2\frac{d\phi}{dt} = r^2\frac{d\phi}{dr}\frac{dr}{dt}$$

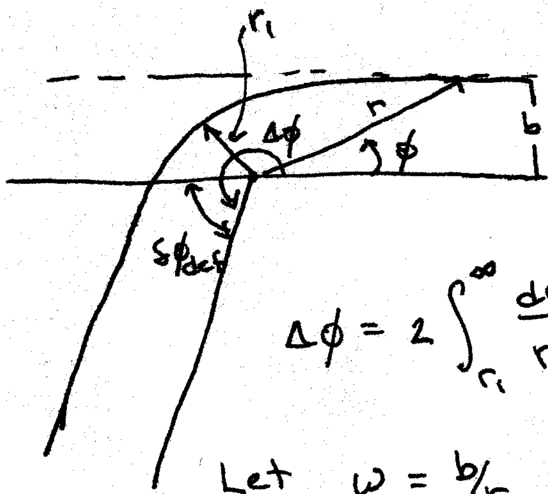
$$\approx r^2\left(-\frac{d}{r^2}\right)(-1) = d$$

<u>b is the impact parameter</u> at ∞.

The shape of the orbit is

$$\frac{d\phi}{dr} = \frac{d\phi/d\lambda}{dr/d\lambda} = \pm\frac{1}{r^2}\left[\frac{1}{b^2} - W_{eff}\right]^{-1/2}$$

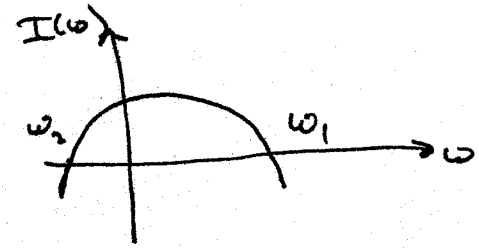
Once again,



$$\Delta\phi = 2\int_{r_1}^{\infty} \frac{dr}{r^2}\left[\frac{1}{b^2} - \frac{1}{r^2}\left(1-\frac{2M}{r}\right)\right]^{-1/2}$$

Let $\omega = b/r$ or $r = b/\omega$ then

$$\Delta\phi = 2\int_0^{\omega_1} d\omega \left[1-\omega^2\left(1-\frac{2M}{b}\omega\right)\right]^{-1/2}$$



This integral has the same form as above! To first order in M/b we get,

$$\Delta\phi = \pi + \frac{4M}{b}$$

Then $\delta\phi_{def} = \Delta\phi - \pi = \boxed{\frac{4M}{b}} = \boxed{\frac{4GM}{bc^2}}$

For bending around the sun

$$b \simeq R_\odot = 6.96\times10^5\,\text{km}, \quad M_\odot = 1.47\,\text{km}$$

$2M/b$ is small! So expand in it,

$$\Delta\phi = 2\int_0^{\omega_1} d\omega \left(1-\frac{2M}{b}\omega\right)^{-1/2}\left[\left(1-\frac{2M}{b}\omega\right)^{-1} - \omega^2\right]^{-1/2}$$

$$\simeq 2\int_0^{\omega_1} d\omega \, \frac{1+\frac{M}{b}\omega}{\sqrt{\underbrace{1+\frac{2M}{b}\omega-\omega^2}_{I(\omega)}}}$$

For starlight grazing the sun this is,

$$\delta\phi_{def} = 1.7''$$

Concludes intro to Schwarzschild and tests of G.R.

Skip ch 10 – more detail on tests

Skip ch 11 – Lensing, accretion, pulsars

Begin Black Holes with ch 12