Today's Outline

Lecture 15

Mar 8th, 2012

I Schwarzschild Black Hole

II Light cones

III Event Horizon

0. Announce Strominger talk on Monday

According to Newton's theorem a time dependent mass distribution $\rho_m(t)$, if spherically symmetric, yields a potential that only depends on $M = \int \rho_m(t_0)\, d^3x$,

$$V = -\frac{GM}{r}.$$

There is a corresponding thm in G.R called Birkhoff's thm. (of course, this is outside region containing masses)

I Sch. Black Hole

When a star runs out of fuel there are two possibilities:

- The remaining mass is supported by non-thermal pressures (e.g. Fermi pressure's).
- No available pressure can support the remaining mass. The star collapses.

Let's assume spherical collapse

Thus outside the (spherical) collapse the spacetime only depends on M and is in fact Sch.

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

Now, however, we should think more carefully about $r=0$ and $r=2M$.

<u>$r=2M$ is a coord. singularity</u>

We can demonstrate that a singularity

is a coord. sing. by finding any coord.s in which it does not appear.

The problem term here is the coeff. of dr^2 and we can design coord.s to get rid of it: Eddington - Finkelstein coord.s $(t, r, \theta, \phi) \to (v, r, \theta, \phi)$ by

$$t = v - r - 2M \log\left|\frac{r}{2M} - 1\right|$$

Important Calculation:

For $r > 2M$

$$dt = dv - dr - 2M \frac{1}{\frac{r}{2M} - 1} \cdot \frac{dr}{2M}$$

$$= dv - \left(1 + \frac{1}{\frac{r}{2M} - 1}\right) dr$$

$$= dv - \left(\frac{r/2M}{r/2M - 1}\right) dr$$

$$= dv - \left(1 - \frac{2M}{r}\right)^{-1} dr$$

$$\Rightarrow dt^2 = dv^2 + \left(1 - \frac{2M}{r}\right)^{-2} dr^2 - 2\left(1 - \frac{2M}{r}\right)^{-1} dv\,dr$$

Then,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + \left(1 - \frac{2M}{r}\right) dr^2 + 2\,dv\,dr + r^2 d\Omega^2$$

$$= -\left(1 - \frac{2M}{r}\right) dv^2 + 2\,dv\,dr + r^2 d\Omega^2$$

No singularity at $r = 2M$! This demonstrates that $r = 2M$ was always just a coordinate singularity. By contrast, we will eventually show that the curvature goes to ∞ at $r = 0$ and so no coord.s can remove the singularity at $r = 0$.

II Light cones

To simplify focus on radial light rays for which $d\theta = d\phi = 0$. Can find light rays from

$$ds^2 = 0 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2\,dv\,dr$$

Three types of solution:

(i) $dv = 0 \Rightarrow v = \text{const.}$

To keep v const. while t increases, r must decrease $\Rightarrow$ these are radially infalling light rays.

(ii) $r = 2M \Rightarrow dr = 0$ and coeff. dv is zero.

Stationary light rays, stuck at $r = 2M$.

(iii) $dv \neq 0 \Rightarrow -\left(1 - \frac{2M}{r}\right) dv + 2\,dr = 0$

$$t = r + 2M \log\left|\frac{r}{2M} - 1\right| + \text{const.}$$

and for increasing t we have two cases

outgoing when $r > 2M$

ingoing when $r < 2M$.

Let's plot 'em: let $\tilde{t} = v - r$ then rays of type (i) are 45° lines with $\tilde{t} = -r + \text{const.}$

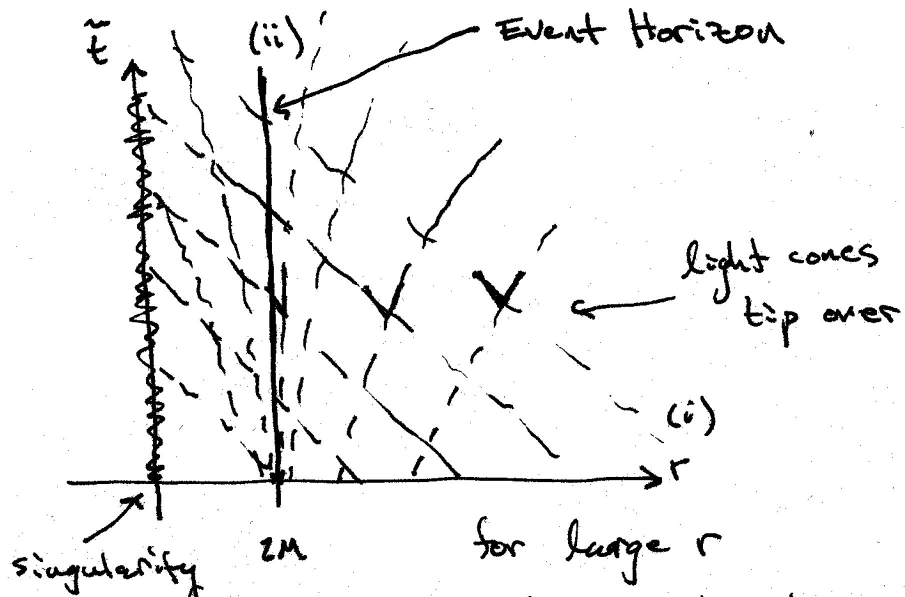
or

$$dv = \frac{2\,dr}{\left(1 - \frac{2M}{r}\right)}$$

But we just took the derivative of something that gave us $\left(1 - \frac{2M}{r}\right)^{-1}$, so

$$v = 2\left(r + 2M \log\left|\frac{r}{2M} - 1\right|\right) + \text{const.}$$

How do these behave (outgoing or ingoing)? Well, for these solns



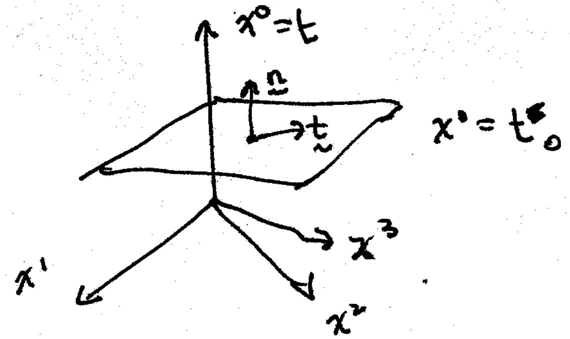
for large r log is small and

$$t = r + \text{const}$$

$\Rightarrow$ 45° far away

III The Event Horizon

A 3-surface is a surface determined by 1 condition on our 4 coordinates, e.g. $t =$ const.



More generally,

$f(x^\alpha) = 0$ expresses the 1 condition.

The normal to a 3-surface is ~~the~~ a vector $\underset{\sim}{n}$ that is orthogonal to all the tangent vectors of the surface,

$$\underset{\sim}{n} \cdot \underset{\sim}{t} = 0$$

We call a surface

spacelike if $\underset{\sim}{n}$ is timelike

timelike if $\underset{\sim}{n}$ is spacelike

null if $\underset{\sim}{n}$ is null

Null surfaces are a bit odd because if $\underset{\sim}{n}$ is null then it's orthogonal to itself

$$\underset{\sim}{n} \cdot \underset{\sim}{n} = 0$$

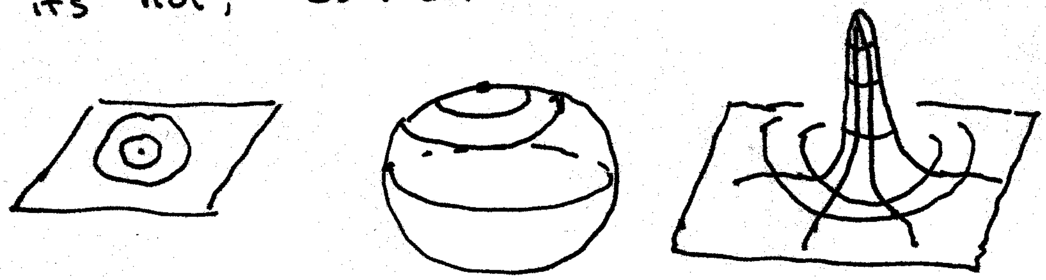
So, for example, if we have ~~we can choose~~ a null surface with three tangent vectors $\underset{\sim}{t_1}, \underset{\sim}{t_2}, \underset{\sim}{l}$

s.t. $\underset{\sim}{t_1} \cdot \underset{\sim}{l}, \underset{\sim}{t_2} \cdot \underset{\sim}{l} = 0$

then we can choose $\underset{\sim}{l}$ as our normal vector to the surface.

The event horizon is a null 3-surface given by $r = x^1 = 2M$.

[Aside on Schw. radius r: You may have noticed that I've been careful not to refer to r as the radial distance from a fixed center. That's because it's not, Consider



For highly curved surfaces the geodesic distance from the center of curvature can be quite large, while most of the action occurs "near" to the peak curvature. By using $r \equiv \frac{\text{Circ}}{2\pi}$ we avoid the large geodesic distances. The Schw. radius r is similar, it's called an areal radius because a surface at fixed t and $r = r_0$ has surface area

$$4\pi r_0^2.$$

End aside]

The area of the event horizon is given by integrating a $v =$ const. slice of it

$$d\Sigma^2 = (2M)^2 (d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$\Rightarrow \quad A = 16\pi M^2$$

Because Schw. is static this is independent of v (and t).

<u>The sign switch</u>

Inside $r = 2M$ note that

$$ds^2 = \underbrace{-\left(1 - \frac{2M}{r}\right)}_{\text{is positive}} dv^2 + 2\,dv\,dr + r^2 d\Omega^2$$

This means that a 3-surface of constant r has line elem.

$$dS^2 = \underbrace{-\left(1 - \frac{2M}{r}\right)}_{\text{positive}} dv^2 + r^2 d\Omega^2$$

This is a spacelike surface! Inside the horizon r acts like a time coordinate and the $r = 0$ singularity is not a place in space but rather a moment in time!