Today's Outline

I Accelerated Worldlines

II Coordinates for an accelerated observer

Lecture 18

Mar 20th, 2012

I Accelerated Worldlines

Consider the worldline

$$t(\sigma) = a^{-1}\sinh\sigma \quad x(\sigma) = a^{-1}\cosh(\sigma)$$

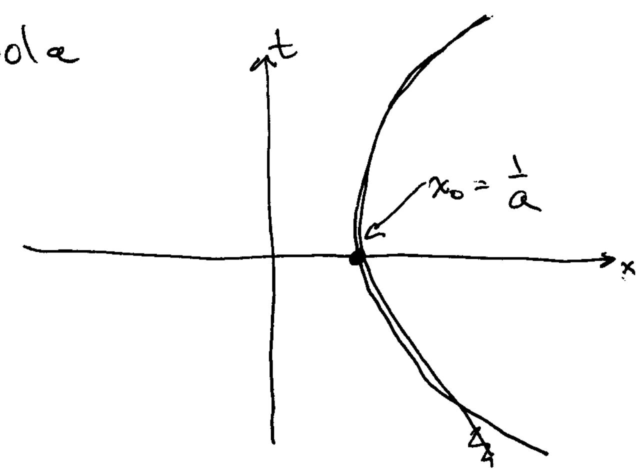
(we'll suppress y and z for now).

What is this worldline? Well,

$$x^2 - t^2 = a^{-2}\left(\cosh^2(\sigma) - \sinh^2(\sigma)\right)$$

$$= a^{-2}$$

So geometrically it is a hyperbola



Certainly a particle following this worldline is accelerated but how? Let's find τ,

$$d\tau^2 = -ds^2 = dt^2 - dx^2$$

And,

$$\frac{d}{d\sigma}(\sinh\sigma) = \frac{d}{d\sigma}\left(\frac{e^{\sigma} - e^{-\sigma}}{2}\right)$$

$$= \frac{e^{\sigma} + e^{-\sigma}}{2} = \cosh\sigma$$

So,

$$d\tau^2 = a^{-2}\cosh^2\sigma\, d\sigma^2 - a^{-2}\sinh^2\sigma\, d\sigma^2$$

$$= (a^{-1} d\sigma)^2$$

For $\tau = 0$ when $\sigma = 0$ we get,

$$\tau = \sigma/a$$

So, in terms of τ this world line is,

$$t(\tau) = a^{-1}\sinh(a\tau) \qquad x(\tau) = a^{-1}\cosh(a\tau)$$

But then,

$$\frac{dx}{d\tau} = u^x = \sinh(a\tau)$$

and

$$\frac{d^2x}{d\tau^2} = a^x = a\cosh(a\tau)$$

A reasonably simple acceleration in the inertial tx-frame.

Then in this frame,

$$\begin{pmatrix} a^t \\ a^x \\ a^y \\ a^z \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ & & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ a \\ 0 \\ 0 \end{pmatrix}$$

So that,

$$a^t = \beta\gamma a$$

$$a^x = \gamma a$$

But $a^t = \frac{d^2t}{d\tau^2} = \frac{d}{d\tau}(\gamma)$ and so,

$$\frac{d\gamma}{d\tau} = \beta\gamma a$$

How does this relate to your homework problem?

You solved the problem for which an instantaneously co-moving observer measured,

$$a^\mu = (0, a, 0, 0)$$

To make sense of this you had to refer everything back to a single inertial frame, say the tx-frame

Also, $\gamma = \frac{1}{\sqrt{1-\beta^2}} \Rightarrow \frac{1}{\gamma^2} = 1-\beta^2$

$$\Rightarrow \beta^2 = 1 - \frac{1}{\gamma^2}$$

Then,

$$\frac{d\gamma}{d\tau} = \sqrt{\gamma^2-1}\; a$$

or

$$\int \frac{d\gamma}{\sqrt{\gamma^2-1}} = a\tau + C$$

$$\Rightarrow \cosh^{-1}(\gamma) = a\tau + C \Rightarrow \gamma = \cosh(a\tau + C)$$

We have $\gamma(\tau=0)=1$ for a particle starting from rest, so,

$$\gamma = \cosh(a\tau)$$

Then

$$\frac{dt}{d\tau} = \gamma = \cosh(a\tau) \Rightarrow t = a^{-1}\sinh(a\tau)$$

From second equation,

$$\frac{du^x}{d\tau} = \cosh(a\tau)\,a$$

$$\Rightarrow u^x = \sinh(a\tau)$$

$$\Rightarrow x = \frac{\cosh(a\tau)-1}{a} + x_0$$

for $x(\tau=0)=x_0$. Or

$$x = a^{-1}\cosh(a\tau) + (x_0 - \tfrac{1}{a})$$

Notice that $\gamma = \cosh(a\tau)$ and $\beta\gamma = \sinh(a\tau)$ allows us to identify the Lorentz transformation made at the beginning of this problem,

$$\begin{pmatrix} a^t \\ a^x \\ a^y \\ a^z \end{pmatrix} = \begin{pmatrix} \cosh(a\tau) & \sinh(a\tau) & 0 & 0 \\ \sinh(a\tau) & \cosh(a\tau) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ a \\ 0 \\ 0 \end{pmatrix}$$

This connects with our midterm problem on Lorentz transf.s as hyperbolic rotations; A uniformly accelerated observer is reached by a simple τ dependent Lorentz transformation!

II Coordinates for an accelerated observer

We now have some experience in building coordinate systems. This proceeds in 2 steps:

(i) Determine the orthonormal frame (like coords at a point)

(ii) Extend this to a small neighborhood using curves.

Let's do it:

(i) We ~~check~~ want to choose

$$\underset{\sim}{e}_{\hat 0}(\tau) = \underset{\sim}{u}_{obs}(\tau)$$

So,

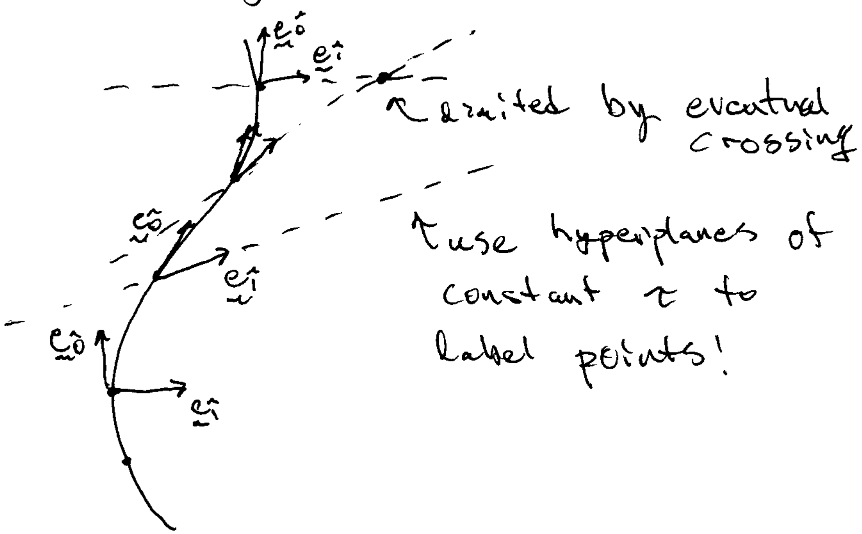
$$(\underset{\sim}{e}_{\hat 0}(\tau))^\alpha = u^\alpha_{obs} = (\cosh(a\tau), \sinh(a\tau), 0, 0)$$

Natural to choose

$$(\underset{\sim}{e}_{\hat 2})^\alpha = (0,0,1,0)$$

$$(\underset{\sim}{e}_{\hat 3})^\alpha = (0,0,0,1)$$

The second step is a bit tricky.
Consider a general acceleration



The last is determined by orthonormality:

$$(\underset{\sim}{e}_{\hat 1}(\tau))^\alpha = (f(\tau), g(\tau), 0, 0)$$

$$\underset{\sim}{e}_{\hat 0} \cdot \underset{\sim}{e}_{\hat 1} = 0 \Rightarrow$$

$$-\cosh(a\tau)\, f(\tau) + \sinh(a\tau)\, g(\tau) = 0$$

$$\underset{\sim}{e}_{\hat 1} \cdot \underset{\sim}{e}_{\hat 1} = -f^2 + g^2 = 1$$

A solution is,

$$(\underset{\sim}{e}_{\hat 1}(\tau))^\alpha = (\sinh(a\tau), \cosh(a\tau), 0, 0)$$

A point in the hyperplane is given by

$$\underset{\sim}{x} = X\, \underset{\sim}{e}_{\hat 1}(\tau) + Y\, \underset{\sim}{e}_{\hat 2}(\tau) + Z\, \underset{\sim}{e}_{\hat 3}(\tau) + \underset{\sim}{\bar{x}}(\tau)$$

position on world line.

and $T = \tau$.

Do it for our uniform case

$$x^0 = a^{-1}\sinh(aT) + X\sinh(aT)$$

$$x^1 = a^{-1}\cosh(aT) + X\cosh(aT)$$

$$x^2 = Y, \quad x^3 = Z$$

This is precisely the coordinate transformation you made in HW #3 prob. 3 (Hartle 6.6) and you showed that it was consistent with Newtonian gravity (in non-rel. limit) and weak-field-gravity time dilation.

What is the resulting metric?

A little algebra shows that it is,

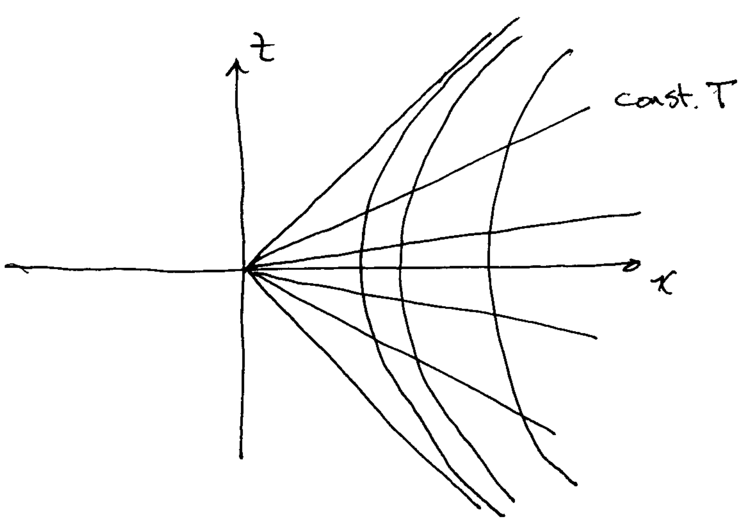
$$ds^2 = -(1+aX)^2(dT)^2 + dX^2 + dY^2 + dZ^2$$

There are several similarities between the capital (T, X, Y, Z) coord.s and Kruskal coord.s. Note, for example, that

$$\frac{x^0}{x^1} = \tanh(aT)$$

so lines of constant T are straight lines in the $x^0 x^1 = tx$-plane

while, by definition, lines of constant X are hyperbolae.



See Eugenio's talk on bspace for more on this topic.

The limiting cases $x^2 - t^2 = 0$ or $x = t$ and $x = -t$ are one way membranes for light signals. Then it makes sense to say that Rindler Spacetime (the name we give to flat Spacetime in these coord.s) has a horizon!