Lecture 19

Mar 22^{nd}, 2012

Today's Outline

I Last Lecture

II Why gyros?

III Motion of gyros

IV Geodetic Precession

V Gyros and Rotating Sources

I Last Lecture

- Reconsidered acceleration using only special relativity.
- Reviewed how to construct $\underset{\sim}{e}_{\hat{\alpha}}$: pick $\underset{\sim}{e}_{\hat{0}} = \underset{\sim}{u}_{obs}$ and use orthonormality, $\underset{\sim}{e}_{\hat{i}} \cdot \underset{\sim}{e}_{\hat{0}} = 0$ $\underset{\sim}{e}_{\hat{i}} \cdot \underset{\sim}{e}_{\hat{i}} = 1$ e.g.
- Saw that accelerated observers find precisely a spacetime that looks like part of Schw. in Kruskal coordinates: the equivalence principle.

II Why gyros?

Atleast ~~two~~ three reasons:

(i) Conceptual: Much like the identification $\underset{\sim}{u}_{obs} = \underset{\sim}{e}_{\hat{0}}$ gives an operational way to determine $\underset{\sim}{e}_{\hat{0}}$, (simply wear a wristwatch and you can monitor how your coordinates change with proper time, τ.) gyros give an operational way to determine $\underset{\sim}{e}_{\hat{1}}, \underset{\sim}{e}_{\hat{2}}, \underset{\sim}{e}_{\hat{3}}$. Align three gyros in three mutually orthogonal directions and keep them with you.

(ii) Experimental tests: In today's lecture we will find that G.R. predicts two types of precession, called "geodetic precession" and "frame dragging." Experimental seaches for these effects can distinguish G.R. from other gravitational theories.

(iii) Gravitomagnetism: Investigation of gyros reveals a completely new (to us) aspect of G.R. Not only the presence of mass/energy curves spacetime but also its motion! This is closely analagous to E&M: The

presence of charges ~~creates~~ generates an electric field and the motion of charges generates a magnetic field, hence "gravitomagnetism"!

III Motion of gyros

test gyro (spin): A small test body with spin, i.e. angular momentum. Like other free particles, a free gyro follows a geodesic:

$$\frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma = 0,$$

$\underset{\sim}{u}$ the 4-velocity of the gyro.

However, as it moves along this geodesic we also have to describe the evolution of its spin ang. mom. 4-vector $\underset{\sim}{S}(\tau)$. We can evaluate $\underset{\sim}{u} \cdot \underset{\sim}{S}$ by going into the rest frame of the gyro:

$$(\underset{\sim}{u})^\alpha = (1, 0, 0, 0)$$

and

$$(\underset{\sim}{S})^\beta = (0, \vec{S})$$

so,

$$\boxed{\underset{\sim}{u} \cdot \underset{\sim}{S} = 0}$$

Motion will preserve this cndtn as well as (HW)

$$\boxed{S_* = (\underset{\sim}{S} \cdot \underset{\sim}{S})^{1/2} = \text{const.}}$$

thus $\underset{\sim}{S}$ precesses.

In a LIF or flat spacetime we have the E.O.M.

$$\frac{dS^\alpha}{d\tau} = 0 \qquad (\text{LIF or flat})$$

Two principles and an assumption allow us to guess the generalization to G.R. (1) Should recover the above eq. in ~~an~~ LIF (2) Eq. of Motion should take same form in all coordinate systems. (3) Assumption: should be linear in S^α, so all spins ~~behave~~ precess same way

(like Larmour precession $\omega = \frac{gB}{2M}$, independent of $\vec{J}$, the angular momentum). Then the E.O.M is,

$$\boxed{\frac{dS^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma} S^\beta u^\gamma = 0}$$

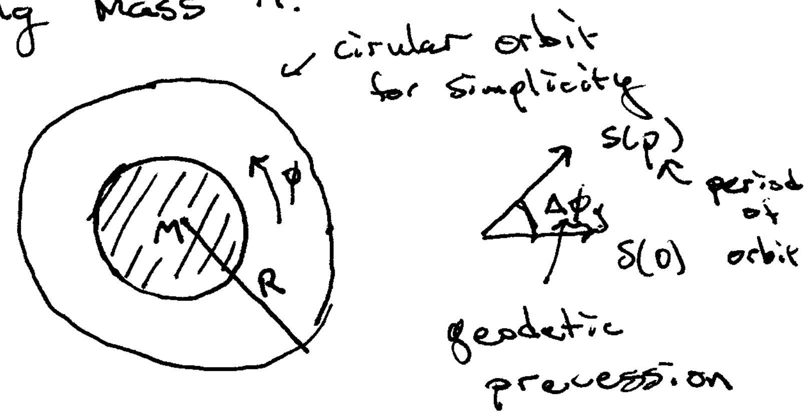
the Gyroscope Equation.

~~On HW, you'll confirm that~~

We'll use this EOM to derive the precession effects mentioned above.

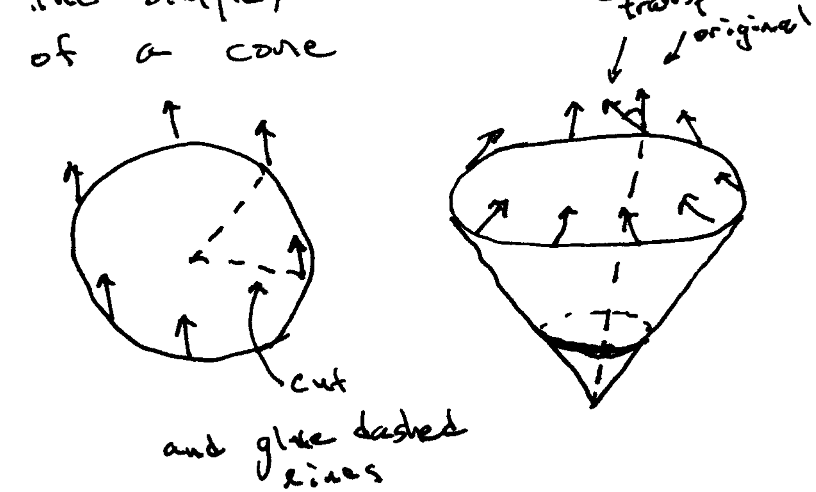
IV Geodetic Precession

This is the precession that occurs upon transporting a gyro around a closed geodesic orbit outside a non-rotating mass M.



Hartle doesn't say why this happens. Two reasons:

- Curved geometry: consider the simple, 2D curved geometry of a cone



The curved geometry causes the gyro to precess upon going around a closed curve! $\frac{2}{3}$ of total precession.

- spin-orbit effect: From the gyro's perspective the mass M is moving $\Rightarrow$ gravitomagnetic effect causes another $\frac{1}{3}$ of total precession.

We'll calculate them together:

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

Circular orbit with $r=R$ and $\theta = \pi/2$, then

$$u^\alpha = (u^t, 0, 0, u^\phi)$$

and
$$u^\phi = \frac{d\phi}{d\tau} = \frac{d\phi}{dt}\frac{dt}{d\tau} = \Omega u^t$$

so,
$$u^\alpha = u^t(1, 0, 0, \Omega),$$

with $\Omega^2 = M/R^3$ (Lecture 13).

Take $(\underset{\sim}{S}\cdot\underset{\sim}{S})^{1/2} = S_*$ and $\underset{\sim}{S}^{\hat{\alpha}}(0) = (0, S_*, 0, 0)$ initially then $S^\theta = 0$ always, due to symmetry and your HW, while,

from $\underset{\sim}{S}\cdot\underset{\sim}{u} = 0$ we find,

$$g_{\alpha\beta}S^\alpha u^\beta = -\left(1-\frac{2M}{R}\right)S^t u^t + R^2 S^\phi u^t \Omega = 0$$

$$\Rightarrow \quad S^t = \left(1-\frac{2M}{R}\right)^{-1} R^2 \Omega S^\phi$$

Use gyro eq. to find S^r and S^ϕ:

$$\frac{dS^r}{d\tau} + \Gamma^r_{tt} S^t u^t + \Gamma^r_{rr} S^r \cancelto{0}{u^r} + \Gamma^r_{\theta\theta} \cancelto{0}{S^\theta} u^\theta + \Gamma^r_{\phi\phi} S^\phi u^\phi = 0$$

$$\Rightarrow \frac{dS^r}{dt}u^t + \frac{M}{R^2}\left(1-\frac{2M}{R}\right)\cdot\left(1-\frac{2M}{R}\right)^{-1} R^2 \Omega S^\phi u^t + [-(R-2M)] S^\phi \Omega u^t = 0$$

$$\Rightarrow \frac{dS^r}{dt} + M\Omega S^\phi + (2M-R)\Omega S^\phi = 0$$

$$\Rightarrow \frac{dS^r}{dt} - (R-3M)\Omega S^\phi = 0 \qquad (1)$$

Similarly,

$$\frac{dS^\phi}{dt} + \frac{\Omega}{R} S^r = 0 \qquad (2)$$

Combining (1) and (2)

$$\frac{d^2 S^\phi}{dt^2} + \left(1-\frac{3M}{R}\right)\Omega^2 S^\phi = 0$$

A harmonic oscillator (!) with freq.
$$\Omega' = \left(1-\frac{3M}{R}\right)^{1/2}\Omega$$

Putting in our initial conditions,

$$s^r(t) = S_* \left(1-\frac{2M}{R}\right)^{1/2} \cos(\Omega' t)$$

$$s^\phi(t) = -S_* \left(1-\frac{2M}{R}\right)^{1/2} \left(\frac{\Omega}{\Omega' R}\right) \sin(\Omega' t)$$

Initially $\underset{\sim}{s}$ pointed along $(\underset{\sim}{e}_{\hat{r}})^\alpha = (0, (1-\frac{2M}{r})^{1/2}, 0, 0)$ (No need to boost into comoving frame because motion is transverse to $\underset{\sim}{e}_{\hat{r}}$.)

so after a period $P = \frac{2\pi}{\Omega}$ we have,

$$\left[\frac{\underset{\sim}{s}(t)}{S_*}\cdot \underset{\sim}{e}_{\hat{r}}\right]\Bigg|_{t=P} = \cos(\Omega' P) = \cos\left(\frac{2\pi\Omega'}{\Omega}\right)$$

$$= \cos\left(2\pi\left(1-\frac{3M}{R}\right)^{1/2}\right)$$

Note that for small M/R this has a leading 2π, so

$$\boxed{\Delta\phi_{\text{geodetic}} = 2\pi\left[1-\left(1-\frac{3M}{R}\right)^{1/2}\right]}$$

$$\Delta\phi_{\text{geodetic}} \approx 6.5\times 10^{-8}\left(\frac{R_\oplus}{R}\right)\text{rad}$$

$R_\oplus$: 6,378 km

$$\longrightarrow 8.4''\left(\frac{R_\oplus}{R}\right)^{5/2} \text{ per year}$$

V Gyros and Rotating Sources

If the massive source is rotating it gives rise to a different (from Schwarzschild) spacetime geometry:

$$ds^2 = ds^2_{\text{Schw}} - \frac{4GJ}{c^3 r^2}\sin^2\theta\,(r\,d\phi)(c\,dt)$$

$$+ \left(\text{terms quadratic and higher order in } J\right)$$

J: magnitude of angular momentum of the source

Gravitomagnetism more precisely:

S.R. scale $\quad \frac{V}{c}$

G.R. scale $\quad \frac{GM}{Rc^2}$

Gravitomag Scale $\quad \frac{GJ}{R^2c^3}$

Estimate: $J \sim I\Omega \sim MR^2\Omega \sim MRV$

So,

$$\frac{GJ}{R^2c^3} \sim \frac{GM}{Rc^2}\cdot\frac{V}{c}$$

$\frac{GM}{Rc^2}$: G.R. effect

$\frac{V}{c}$: moving

} weaker i.e. higher order effect.

This metric causes a precession independent of the geodetic effect. See book for derivation,

$$\Omega_{LT} = \frac{2GJ}{c^2 z^3}$$

Lense-Thirring precession

A.K.A. "frame dragging".

These two effects have recently been measured by Gravity Probe B

	Geodetic	Frame drag
G.R.	$-6{,}606.1$ mas/yr	-39.2 mas/yr
GPB	$-6{,}601.8 \pm 18.3$ mas/yr	-37.2 ± 7.2 mas/yr