Lecture 2

Jan. 19th, 2012

Today's Outline:

O. Announcements

I. What is geometry?

II. Lagrangian Mechanics

O. Announcements

- 2nd Discussion Section:

  Mon. 11-12pm 2070 VLSB

  ↳ sorry - scheduling issue

  CCN# 69530

  1st Disc. Section Today 4-5pm in 70 Evans, but presumably it will also move.

- Mathematica Blunder
- No Off. Hours for Hal this Friday, Jan. 20th.
- Course Plan: Treat Special Relativity Briefly → focus on G.R.

Results of Survey: 1st: Black Holes

2nd: Gravitational Waves

Distant 3rd: Cosmology

Plan: Foundations Ch.s 1-9

B.H.s Ch.s 12-15

Grav. Waves & Field Eqns

Ch 16 & Ch.s 20-21

Exercise: Figure out Cosmology

I What is geometry?

Study of shape, size, position and the properties of space.

Discuss two revolutions:

$\boxed{\text{There exists more than one geometry!}}$

Examples from 2D: In the usual Euclidean plane we have,

$$\sum_{\substack{\text{vertices}\\ \text{in tri}}} \begin{pmatrix}\text{interior}\\ \text{angle}\end{pmatrix} = \pi$$

and,

$$\frac{\text{Circum.}}{\text{radius}} = \frac{c}{r} = 2\pi$$

Consider instead the surface of a sphere. How do we even define a triangle? A figure consisting of three "~~straightest~~ straight as possible" sides.

"Straight as possible" determined by shortest path joining end points; called <u>great circles</u>. Spherical triangles satisfy

$$\sum_{\text{vertices}} \begin{pmatrix}\text{int.}\\ \text{angle}\end{pmatrix} = \pi + \frac{\text{Area}}{a^2}$$

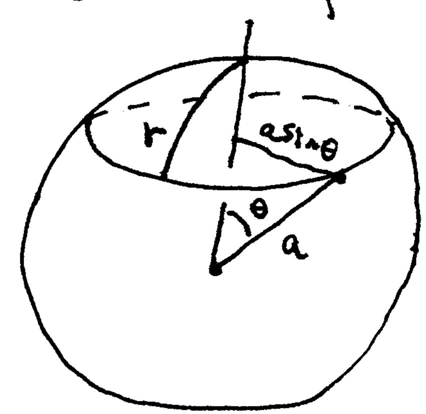
radius of sphere a

e.g.

$$\sum_{\text{vertices}} \begin{pmatrix}\text{int.}\\ \text{ang.}\end{pmatrix} = \frac{3\pi}{2} = \pi + \frac{\frac{1}{8}(4\pi a^2)}{a^2} = \frac{3\pi}{2} \checkmark$$

Note: Can generate a great circle using a plane through the origin.

Another example



Here,

$$C = 2\pi\, a \sin\theta \qquad \overset{\text{arclength}}{r} = a\,\theta$$

radius, angle

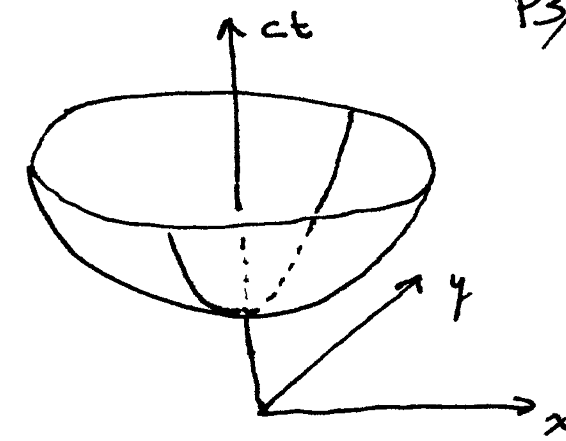
$$= 2\pi\, a \sin(r/a)$$

so that

$$\frac{C}{r} = 2\pi\, \frac{\sin(r/a)}{r/a}$$

Another example: Anticipating S.R.

consider a space w/ coord.s (ct, x, y)



and the surface

$$-(ct)^2 + x^2 + y^2 = -1$$
$$ct > 0$$

or $ct = +\sqrt{1 + x^2 + y^2}$

called the upper sheet of the two sheeted hyperboloid.

"Straight as possible" lines can again be generated by planes through the origin,

Hyperbolic triangles satisfy

$$\sum_{\text{vertices}} \binom{\text{int.}}{\text{angle}} = \pi - \frac{\text{Area}}{R^2}$$

constant radius of curvature of hyperboloid

These are just two examples – very special examples in fact, they both have constant curvature. – others would include the surface of an egg, peanut or football.

Already it's clear how limited our calculational techniques are.

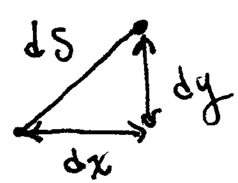
2nd Revolution (Riemann):

> Characterize geometry locally and allow variation from point to point.

Main approach: Specify the infinitessimal distance between neighboring points, dS. Use integral and differential calculus to build up finite distance, areas, angles, etc.

A surface's extrinsic geometry describes how it sits inside a larger space.

Ex.: Euclidean plane. Introduce coords (x,y)

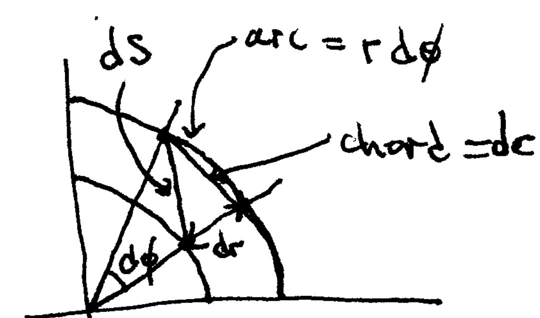


$$dS^2 = dx^2 + dy^2$$

or coord.s (r, ϕ)

convention: Capital 'S' for 2D and 3D, lower case 's' for 4D.

- Important insight: This allows one to characterize a geometry <u>intrinsically</u>, that is, without viewing it "from the outside" as a surface (or hyper surface) in a higher dimensional space.



$$dS^2 = dr^2 + dc^2$$

$$dc = 2(r+dr)\sin\left(\frac{d\phi}{2}\right)$$

$$\simeq 2r\frac{d\phi}{2} + O(dr\,d\phi,\ d\phi^3, \ldots)$$

$$\Rightarrow dS^2 = dr^2 + r^2 d\phi^2$$

Check that you get the same formula $dS(r, \phi)$ by doing a coordinate transformation of $dS(x,y)$; this is important, we don't want our physical results to depend on our choice of coord.s

II Lagrangian Mechanics

$T = K.E.$, $\quad V = P.E.$

E.O.M. : $\quad \frac{d}{dt}(m\dot{x}) = m\ddot{x} = -\frac{\partial V}{\partial x} = F$ ✓

IF several coord.s : $L(x^i, \dot{x}^i, t)$ $(i = 1, \dots, n)$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{\partial L}{\partial x^i}} \quad (i = 1, \dots, n)$$

Hamilton's Principle (Variational Prin.)

The actual path taken is the one that minimizes the Action.

Lagrangian: $L = T - V$

$$L = L(x, \dot{x}, t), \quad \dot{x} = \frac{dx(t)}{dt}$$

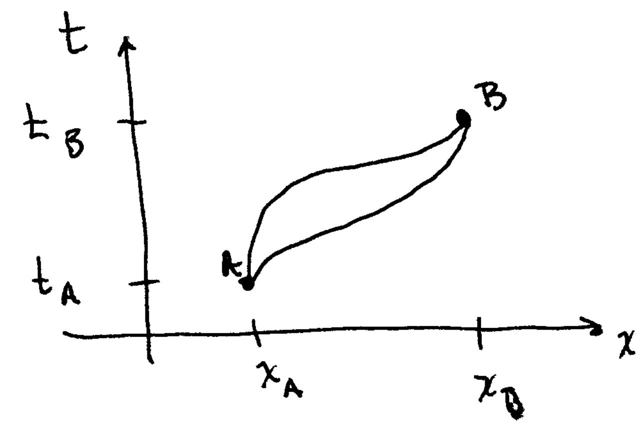
Equations of motion:

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}}$$

Ex. $\quad T = \frac{1}{2} m \dot{x}^2 \quad V = V(x)$

$$\Rightarrow \quad L = \frac{1}{2} m \dot{x}^2 - V(x)$$

$\partial L / \partial \dot{x} = m\dot{x} \quad \partial L / \partial x = -\frac{\partial V}{\partial x} = F$



Action:

$$S[x(t)] = \int_{t_A}^{t_B} dt\, L(x(t), \dot{x}(t), t)$$

"Functional" : functions $\rightarrow$ numbers

For the physical path $\boxed{\delta S = 0}$.

Computing,

$$\delta S = \int_{t_A}^{t_B} \left(\frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} \right) dt$$

Why is it that

$$\delta \left(\frac{dx}{dt} \right) = \frac{d}{dt} (\delta x) \ ?$$

Well,

$x(t)$ (true path)

$x(t) + \epsilon(t)$ (varied path)

$$\Rightarrow \quad \delta x = \epsilon(t)$$

$$\delta \dot{x} = \delta (\dot{x} + \dot{\epsilon}) = \dot{\epsilon} = \frac{d\epsilon}{dt} = \frac{d}{dt}(\delta x)$$

Back to,

$$\delta S = \int_{t_A}^{t_B} \left(\frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \frac{d}{dt} \delta x \right) dt$$

integrate by parts

$$= \int_{t_A}^{t_B} \left(\frac{\partial L}{\partial x} \delta x - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} \right) \delta x \right) dt + \left. \frac{\partial L}{\partial \dot{x}} \delta x \right|_A^B$$

But $\delta x(A) = \delta x(B) = 0$,

$$\delta S = \int_{t_A}^{t_B} \left[\frac{\partial L}{\partial x} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) \right] \delta x \, dt$$

$$\delta S = 0 \quad \Rightarrow \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x} \ !$$

In more detail:

$$\delta x \overset{\text{means}}{=} [x(t) + \epsilon(t)] - [x(t)]$$

$$= \epsilon(t)$$

$$\delta \dot{x} \overset{\text{means}}{=} [\dot{x}(t) + \dot{\epsilon}(t)] - [\dot{x}(t)]$$

$$= \dot{\epsilon}(t) = \frac{d}{dt}(\epsilon(t)) = \frac{d}{dt}(\delta x).$$