Lecture 21

April 5th, 2012

Today's Outline:

I Last Lecture

II The Kerr Horizon

III Survey Results

IV The Ergosphere

I Last Lecture

Revision: Some Black Holes

	Non-rotating $J=0$	Rotating $J \neq 0$
Uncharged $Q=0$	Schwarzschild	Kerr
Charged $Q \neq 0$	Reissner-Nordström	Kerr-Newman

Reissner-Nordström: $\vec{E} \neq 0$, $\vec{B} = 0$

Kerr-Newman: $\vec{E} \neq 0$, $\vec{B} \neq 0$!

Kerr Metric:

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 - \frac{4Mar\sin^2\theta}{\rho^2}\,d\phi\,dt + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta\,d\phi^2$$

with $a = \frac{J}{M}$, $\rho^2 = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - 2Mr + a^2$

- Noted stationary and ϕ indep.

$\Rightarrow \xi^\alpha = (1,0,0,0)$ and $\eta^\alpha = (0,0,0,1)$

- Guessed that $\Delta = 0$ was a coord. singularity and that $r_\pm$ represented event horizons:

$$r_\pm = M \pm \sqrt{M^2 - a^2}$$

while $\rho = 0$ represented a true curvature singularity.

- Noted various limits

$a \to 0$ recover Schwartschild

$r \gg M, a$ Asymptotically flat

$M \to 0$ Recover Mink. Spacetime.

II The Horizon: We can check our guess about whether r_+ is an event horizon by checking if it is a null surface.

Recall a null surface has 3 tangent vectors $\underset{\sim}{l}, \underset{\sim}{t_1}, \underset{\sim}{t_2}$ s.t.

$$\underset{\sim}{l}\cdot\underset{\sim}{l}=0 \quad \underset{\sim}{l}\cdot\underset{\sim}{t_1}=0 \quad \underset{\sim}{l}\cdot\underset{\sim}{t_2}=0 \quad \text{etc.}$$

These are tangent vectors to a surface of constant r, i.e.

$$t^r = (t^t, 0, t^\theta, t^\phi)$$

The null vector must satisfy

$$\underset{\sim}{l}\cdot\underset{\sim}{l} = g_{tt}(l^t)^2 + 2g_{t\phi}l^t l^\phi + g_{\phi\phi}(l^\phi)^2 + g_{\theta\theta}(l^\theta)^2 = 0$$

if it exists. Now, $g_{\theta\theta} = \rho_+^2 = \rho(r_+,\theta)^2$ is positive and so $l^\theta = 0$ must hold.

Clever algebra manipulation leads to,

$$\left(\frac{2Mr_+\sin\theta}{\rho_+}\right)^2\left(l^\phi - \frac{a}{2Mr_+}l^t\right)^2 = 0$$

and again the positive coeff. implies that

$$\boxed{l^\phi = \frac{a}{2Mr_+}l^t}$$

Then $\exists$ "there exists" a null vector and it is,

$$l^\alpha = (1, 0, 0, \Omega_H)$$

"Arbitrary multiple of this vector is also null!"

with

$$\boxed{\Omega_H \equiv \frac{a}{2Mr_+}}$$

The Horizon light rays rotate w.r.t. infinity!

$$d\phi/dt = \frac{d\phi}{d\lambda}\cdot\left(\frac{dt}{d\lambda}\right)^{-1} = \frac{l^\phi}{l^t} = \Omega_H$$

The horizon is no longer a sphere: at r_+ and $t=$const.

$$d\Sigma = \rho_+^2(\theta)\,d\theta^2 + \left(\frac{2Mr_+}{\rho_+(\theta)}\right)^2\sin^2\theta\,d\phi^2$$

$$A = \iint \rho_+\frac{2Mr_+}{\rho_+}\sin\theta\,d\theta\,d\phi = 4\pi(2Mr_+) = \boxed{8\pi M(M+\sqrt{M^2-a^2})}$$

I found $d\Sigma$ as follows:

$$\Delta = 0 = r_+^2 - 2Mr_+ + a^2 \Rightarrow r_+^2 + a^2 = 2Mr_+$$

$$\rho_+^2 = r_+^2 + a^2(1-\sin^2\theta) = 2Mr_+ - a^2\sin^2\theta$$

So,

$$\left(\frac{2Mr_+\rho_+^2}{\rho_+^2} + \frac{2Mr_+a^2\sin^2\theta}{\rho_+^2} \right)$$

$$= \left(\frac{(2Mr_+)^2 - 2Mr_+a^2\sin^2\theta + 2Mr_+a^2\sin^2\theta}{\rho_+^2} \right)$$

$$= \left(\frac{2Mr_+}{\rho_+(\theta)} \right)^2 \checkmark$$

and we want,

$$\underset{\sim}{u}_{obs} \cdot \underset{\sim}{u}_{obs} = -1 = g_{tt}(u^t_{obs})^2$$

$$= -\left(1 - \frac{2Mr}{\rho^2}\right)(u^t_{obs})^2$$

This condition can only be satisfied when the coeff. of $(u^t_{obs})^2$ is negative!

But at the appropriate r,

$$\left(1 - \frac{2Mr}{\rho^2}\right) = 0$$

$$\Rightarrow r^2 - 2Mr + a^2\cos^2\theta = 0$$

$$\Rightarrow r_e = M + \sqrt{M^2 - a^2\cos^2\theta}$$

III Survey Results

See slides on bspace: resources

Result of vote: Current: 5 2nd option: 9

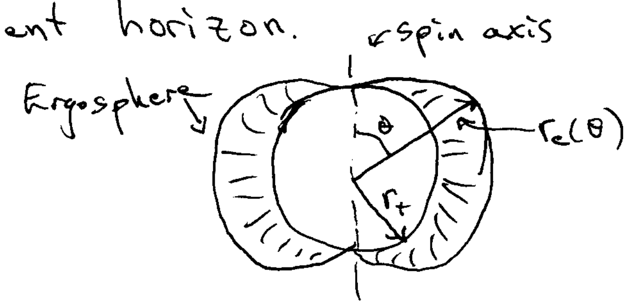
IV The Ergosphere

What kind of $\underset{\sim}{u}$ is required to remain stationary? Well, certainly,

$$u^\alpha_{obs} = (u^t_{obs}, 0, 0, 0)$$

this coefficient vanishes! Inside this surface you can't be a stationary observer, because the coeff > 0; we call it the "ergosphere".

Note that $r_e \geq r_+ = M + \sqrt{M^2 - a^2}$, so the ergosphere is outside the event horizon.



Curvature not accurately depicted (not an embedding diagram)

You can stay at fixed r and θ by rotating with the black hole, e.g. with,

$$u^{\alpha}_{obs} = u^{t}_{obs}\,(1, 0, 0, \Omega_{obs})$$

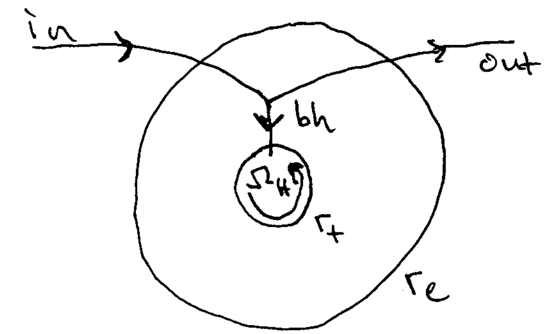
or

$$\underset{\sim}{u}_{obs} = u^{t}_{obs}\,(\underset{\sim}{\xi} + \Omega_{obs}\,\underset{\sim}{\eta})$$

but only a limited range of Ω_{obs} values allow $\underset{\sim}{u}_{obs}$ to be timelike.

The Penrose Process

Schematic at $\theta = \pi/2$,



You can extract energy from a Kerr black hole!

Consider a particle with 4-momentum

$$p^{\mu} = m\,u^{\mu}$$

Its energy and momentum are,

$$E = -\underset{\sim}{\xi}\cdot\underset{\sim}{p} \quad \text{and} \quad L = \underset{\sim}{\eta}\cdot\underset{\sim}{p}$$

(capital letter because they are <u>not</u> per unit mass). Note that

$$\underset{\sim}{\xi}\cdot\underset{\sim}{\xi} = g_{tt} = -\left(1 - \frac{2Mr}{\rho^2}\right)$$

but as we just argued this is positive inside the ergosphere hence $\underset{\sim}{\xi}$ is spacelike. But this means that in this region

$$E = -\underset{\sim}{\xi}\cdot\underset{\sim}{p} \overset{\text{can be}}{<} 0$$

a very strange observation that is slightly more believable when you notice that <u>all</u> particles outside the ergosphere have $E>0$.

Now let's write down conservation of energy-momentum:

$$\underset{\sim}{P}_{in} = \underset{\sim}{P}_{out} + \underset{\sim}{P}_{bh}$$

Dotting with $\underset{\sim}{\xi}$ gives

$$E_{in} = E_{out} + E_{bh}$$

$$\Rightarrow \quad E_{out} = E_{in} - E_{bh}$$

If $E_{bh} < 0$ then $E_{out} > E_{in}$ and

we've extracted energy from the black hole! In fact we can show that the energy we extracted decreases the angular momentum of the black hole:

From above the null generator of the horizon is

$$\underset{\sim}{l} = \underset{\sim}{\xi} + \Omega_H \underset{\sim}{\eta}$$

For a particle to cross the horizon "moving forward in time" we have,

$$\underset{\sim}{P}_{bh} \cdot \underset{\sim}{l} < 0$$

or

$$-E_{bh} + \Omega_H L_{bH} < 0$$

$$\Rightarrow \quad L_{bh} < \frac{E_{bh} \leftarrow \text{negative}}{\Omega_H \leftarrow \text{positive}}$$

We see that L_{bh} is negative, i.e. it's moving against the BH's rotation.

Amazingly you can prove that this does not decrease the area of the event horizon — do it!

The penrose process does not appear to be astrophysically relevant but electromagnetic analogs do!