Today's Outline

I Vectors & Dual Vectors

~~II Tensors~~

Lecture 22

April 10th, 2012

Last time we discussed (i) I relatively informally said: The key is to separate directions & magnitudes. Directions are accesible locally and then we impose linearity i.e.

$$\alpha(\underset{\sim}{a}+\underset{\sim}{b}) = \alpha\underset{\sim}{a}+\alpha\underset{\sim}{b}$$

to build up larger magnitude vectors. We turn now to a careful definition of directions and locally.

I Vectors & Dual Vectors

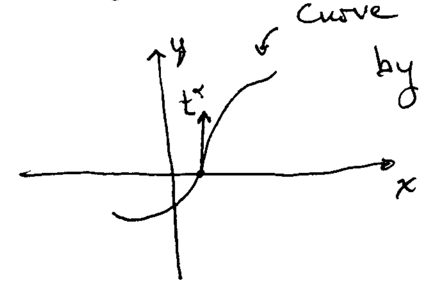
Recall our previous discussion of vectors:

(i) How does a local observer talk about (i.e. measure) vectors?

(ii) Where do vectors live?

(iii) Does this mean that vectors based at different points of our space live in diff. tangent space's?

The definition of directions is, at first, unexpected and later awesome; we use directional derivatives:




Consider $f(x^\alpha(\sigma))$ then the derivative of $f$ in the direction of the curve is:

$$\frac{df}{d\sigma} = \lim_{\epsilon \to 0}\left[\frac{f(x^\alpha(\sigma+\epsilon)) - f(x^\alpha(\sigma))}{\epsilon}\right]$$

$$= \frac{\partial f}{\partial x^\alpha}\frac{dx^\alpha}{d\sigma}$$

The part

$$\frac{dx^\alpha}{d\sigma} = t^\alpha$$

is a vector tangent to the curve at $\sigma$. Specifying the tangent vector specifies the directional derivative,

$$\frac{d}{d\sigma} = t^\alpha \frac{\partial}{\partial x^\alpha} = \frac{dx^\alpha}{d\sigma}\frac{\partial}{\partial x^\alpha}$$

and visa versa, as we've seen in the definition of $\frac{df}{d\sigma}$ above. This also explains what I meant by locally — the whole ~~the~~ construction takes place in the limit $\epsilon \to 0$.

Q: What derivative is associated with $\hat{x} = \underset{\sim}{e}_1$?

The derivative $\partial/\partial x = \frac{\partial}{\partial x^1}$ is a natural guess because it measures changes in the $x^1$-direction and indeed

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x^1} = (0,1,0,0)^\alpha \frac{\partial}{\partial x^\alpha}$$

$$= (\underset{\sim}{e}_1)^\alpha \frac{\partial}{\partial x^\alpha}$$

So this is correct!

Q: More generally what derivative is associated to $\underset{\sim}{e}_\alpha$? Same idea, it's $\partial/\partial x^\alpha$, which we write also as,

$$\underset{\sim}{e}_\alpha = (\underset{\sim}{e}_\alpha)^\beta \frac{\partial}{\partial x^\beta} = \delta^\beta_\alpha \frac{\partial}{\partial x^\beta} = \frac{\partial}{\partial x^\alpha}\,!$$

Q: what does a general vector $\underset{\sim}{a}$ become as a derivative? From the last ~~part~~ question

$$\underset{\sim}{a} = a^\alpha \underset{\sim}{e}_\alpha = a^\alpha \frac{\partial}{\partial x^\alpha}.$$

The tangent space is the linear space of directional derivatives.

A recurring question for us has been "how do objects change when we change coordinates?"

Transformation of vectors:

$$\underset{\sim}{a} = a^\alpha \frac{\partial}{\partial x^\alpha} = a^\alpha \frac{\partial x'^\beta}{\partial x^\alpha}\frac{\partial}{\partial x'^\beta}$$

$$\equiv a'^\beta \frac{\partial}{\partial x'^\beta}$$

So the components in the new coordinates are,

$$\boxed{a'^\beta = \frac{\partial x'^\beta}{\partial x^\alpha} a^\alpha}$$

Similarly

$$a'^\alpha \frac{\partial}{\partial x'^\alpha} = a'^\alpha \frac{\partial x^\beta}{\partial x'^\alpha}\frac{\partial}{\partial x^\beta} = a^\beta \frac{\partial}{\partial x^\beta}$$

so,

$$\boxed{a^\beta = \frac{\partial x^\beta}{\partial x'^\alpha} a'^\alpha}$$

## Dual Vectors

"Sophisticated" definition: "Dual vectors eat vectors and return numbers" or "A dual vector $\underset{\sim}{\omega}$ is a linear map from vectors to real numbers."

Linearity, i.e. $\omega(\alpha \underset{\sim}{a} + \beta \underset{\sim}{b}) = \alpha\,\omega(\underset{\sim}{a}) + \beta\,\omega(\underset{\sim}{b})$,

is a tight restriction, most general such map is

$$\omega(\underset{\sim}{a}) = \omega_\alpha a^\alpha$$

with numbers $\omega_\alpha$, called components.

Physics example: Momentum eats velocity and returns twice the kinetic energy:

$$P(\underset{\sim}{v}) \cancel{= p_\alpha v^\alpha = m v_\alpha v^\alpha}$$

$$= P_i v^i = m v_i v^i = 2\,\text{K.E.}$$

from its def.:

$$P_i = \frac{\partial L}{\partial \dot{q}^i}$$ lower index.

Notice that momentum is naturally dual

Math example: The Gradient of f is a dual vector because,

$$\frac{\partial f}{\partial x^\alpha} t^\alpha = \text{number},$$

it takes $\underset{\sim}{t}$ to a number. We often write it as

$$\underset{\sim}{\nabla} f$$

Dual vectors are linear maps

and so we can also introduce a basis for dual vectors, call it $\{\underset{\sim}{e}^\alpha\}$ $\alpha = 0, 1, 2, 3$. This allows us to decompose,

$$\underset{\sim}{\omega} = \omega_\alpha \underset{\sim}{e}^\alpha .$$

The dual basis vectors $\underset{\sim}{e}^\alpha$ are also maps, they're defined by,

$$e^\alpha(\underset{\sim}{e}_\beta) \equiv \delta^\alpha_\beta \equiv \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}$$

Consistency check:

$$\omega(\underset{\sim}{a}) = \omega_\alpha e^\alpha(a^\beta \underset{\sim}{e}_\beta)$$
$$= \omega_\alpha a^\beta e^\alpha(\underset{\sim}{e}_\beta)$$
$$= \omega_\alpha a^\beta \delta^\alpha_\beta = \omega_\alpha a^\alpha . \checkmark$$

The secret of the metric revealed:

We already have a linear map of vectors defined by $\underset{\sim}{a}$

$$a(\underset{\sim}{b}) = \underset{\sim}{a} \cdot \underset{\sim}{b}$$

But this means $\underset{\sim}{a}$ can be thought of as a dual vector! How?

$$a(\underset{\sim}{b}) = a_\alpha b^\alpha = \underset{\sim}{a} \cdot \underset{\sim}{b} = g_{\beta\alpha} a^\beta b^\alpha$$
$$= (g_{\alpha\beta} a^\beta) b^\alpha$$

So,

$$\boxed{a_\alpha \equiv g_{\alpha\beta} a^\beta}$$

Any vector (e.g. $\underset{\sim}{a}$) can be converted into a dual vector by using the metric to <u>lower</u> an index.

Recall our definition of the inverse metric

$$\boxed{g^{\alpha\gamma} g_{\gamma\beta} = \delta^\alpha_\beta .}$$

We can use the ~~dual~~ inverse metric to raise indices, i.e. convert dual vectors to vectors:

$$\boxed{a^\alpha = g^{\alpha\beta} a_b .}$$

Because of these conversion processes physicists often identify vectors and dual vectors and just speak about the (upper) contra- and (lower) co-variant components of a "vector" (co is down low).

Q: If a vector $\underset{\sim}{a}$ has orthonormal components $a^{\hat{\alpha}}$ what are its lower components? (in same basis).

A: $a_{\hat{\alpha}} = \eta_{\hat{\alpha}\hat{\beta}}\, a^{\hat{\beta}}$

$\Rightarrow a_{\hat{0}} = -a^{\hat{0}}$, $a_{\hat{1}} = a^{\hat{1}}$ etc

Q: How many ways can we write $\underset{\sim}{a}\cdot\underset{\sim}{b}$?

$$\underset{\sim}{a}\cdot\underset{\sim}{b} = g_{\alpha\beta}\, a^{\alpha} b^{\beta} = a_{\beta} b^{\beta} = a^{\alpha} b_{\alpha}$$
$$= g^{\alpha\beta} a_{\alpha} b_{\beta}$$

When we first encountered orthonormal bases, the conversion btwn bases seemed obscure. Now, it's much more obvious,

$$\boxed{a^{\alpha} = \underset{\sim}{e}^{\alpha}\cdot\underset{\sim}{a}, \qquad a_{\alpha} = \underset{\sim}{e}_{\alpha}\cdot\underset{\sim}{a}}$$

E.g.
$$a^{\hat{\alpha}} = \underset{\sim}{e}^{\hat{\alpha}}\cdot\underset{\sim}{a} = \underset{\sim}{e}^{\hat{\alpha}}\cdot\left(a^{\beta}\underset{\sim}{e}_{\beta}\right)$$
$$= a^{\beta}\, \underset{\sim}{e}^{\hat{\alpha}}\cdot\underset{\sim}{e}_{\beta} = a^{\beta}\left(\underset{\sim}{e}^{\hat{\alpha}}\right)_{\beta}$$

<u>Another nice Example</u>: Normal Vectors

Suppose a 3-surface is defined by

$$f(x^{\gamma}) = \text{const.}$$

The gradient of $f$ is normal to this surface,

$$n_{\alpha} = \frac{\partial f}{\partial x^{\alpha}}.$$

This is because, a small displacement $\delta\underset{\sim}{x}$ in the surface doesn't change the value of $f$,

$$\delta f = \left(\frac{\partial f}{\partial x^\alpha}\right) \delta x^\alpha = 0$$

But then,

$$n_\alpha \delta x^\alpha = \underset{\sim}{n} \cdot \delta \underset{\sim}{x} = 0$$

so $\underset{\sim}{n}$ is normal to the surface.

E.g. A sphere $\overset{f(x,y,z)}{x^2+y^2+z^2} = \text{const.} = R^2$

$$\frac{\partial f}{\partial x^i} = (2x, 2y, 2z)$$

which is indeed a radial vector. ✓