Today's Outline:

Lecture 23

April 12th, 2012

I Last Lecture

II Tensors

I Last Lecture

- The tangent space is the linear space of directional derivatives.
- It has a coordinate basis,

$$\underset{\sim}{e}_\alpha = \frac{\partial}{\partial x^\alpha}$$

- A dual vector is a linear map from vectors to real numbers.

- Just like vectors, dual vectors have components. These are just the numbers that result when a dual vector eats a basis vector,

$$\omega(\underset{\sim}{e}_\alpha) = \omega_\beta \, e^\beta(\underset{\sim}{e}_\alpha) = \omega_\beta \, \delta^\beta_\alpha = \omega_\alpha .$$

our definition of dual bases.

- The metric allows us to identify dual vectors and vectors: "raising and lowering indices"

$$\boxed{a_\alpha = g_{\alpha\beta} a^\beta \quad , \quad a^\alpha = g^{\alpha\beta} a_\beta}$$

Q: How many ways can we write $\underset{\sim}{a} \cdot \underset{\sim}{b}$?

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = g_{\alpha\beta} a^\alpha b^\beta = a_\beta b^\beta = a^\alpha b_\alpha$$

$$= g^{\alpha\beta} a_\alpha b_\beta \quad \text{etc.}$$

Final comment: When we first encountered orthonormal bases, the conversion btwn bases seemed obscure. Now, it's much more obvious; just project onto the basis vectors:

$$\boxed{a^\alpha = \underset{\sim}{e}^\alpha \cdot \underset{\sim}{a} \,, \qquad a_\alpha = \underset{\sim}{e}_\alpha \cdot \underset{\sim}{a}}$$

E.g. $a^{\hat\alpha} = \underset{\sim}{e}^{\hat\alpha} \cdot \underset{\sim}{a} = \underset{\sim}{e}^{\hat\alpha} \cdot (a^\beta \underset{\sim}{e}_\beta)$

$$= a^\beta \underset{\sim}{e}^{\hat\alpha} \cdot \underset{\sim}{e}_\beta = a^\beta (\underset{\sim}{e}^{\hat\alpha})_\beta$$

where $(\underset{\sim}{e}^{\hat\alpha})_\beta$ are the coordinate components of the orthonormal frame vectors, as before.

II Tensors

Tensors are unnecessarily shrouded in a fog of confusion: "... baffling beasts bristling with indices ..." The idea of a tensor is actually quite close to what we've just described for dual vectors. A tensor is something that eats vectors (and dual vectors) and returns a number. They do this in a multilinear manner, i.e. linear in each entry. In equations,

$$t(\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}) = t(a^\alpha \underset{\sim}{e}_\alpha, b^\beta \underset{\sim}{e}_\beta, c^\gamma \underset{\sim}{e}_\gamma)$$

$$= a^\alpha b^\beta c^\gamma \, t(\underset{\sim}{e}_\alpha, \underset{\sim}{e}_\beta, \underset{\sim}{e}_\gamma)$$

$$= a^\alpha b^\beta c^\gamma \, t_{\alpha\beta\gamma} \quad \leftarrow \text{components of tensor}$$

We call the number of vectors and dual vectors that a tensor eats its rank (= total # of indices on tensor).

We've been spending lots of time with a particular rank 2 tensor,

$$g(\underset{\sim}{a}, \underset{\sim}{b}) = \underset{\sim}{a} \cdot \underset{\sim}{b} = g_{\alpha\beta} a^\alpha b^\beta$$

the metric!

You can raise and lower indices on a general tensor too — use the metric: Suppose $\underset{\sim}{S}$ eats a vector and two dual vectors

Then,

$$S(\underset{\sim}{a}, \underset{\approx}{\omega}, \underset{\approx}{\lambda}) = S_\alpha{}^{\beta\gamma} a^\alpha \omega_\beta \lambda_\gamma$$

$$= S_{\alpha\beta\gamma} a^\alpha \omega^\beta \lambda^\gamma = S_{\alpha\mu\nu} a^\alpha g^{\mu\beta} \omega_\beta g^{\nu\gamma} \lambda_\gamma$$

So,

$$S_\alpha{}^{\beta\gamma} = S_{\alpha\mu\nu} g^{\mu\beta} g^{\nu\gamma}$$

How do you construct a tensor? There are many ways but one nice example is to do it out of vectors:

$$\gamma^{\alpha\beta\gamma} = u^\alpha v^\beta w^\gamma$$

is a rank 3 tensor

This is called the "outer" or "tensor" product of vectors. (viz. Quantum Mech.)

Q: What is the essential reason this is a tensor?

Because of its manifest multilinearity (recall that vectors eat dual vectors in a linear manner,

$$u(\underset{\sim}{\lambda}) = u^\alpha \lambda_\alpha.)$$

<u>Transformation</u>: suppose we make a coord. transformation from x^α coord.s to x'^β coord.s, what happens to a tensor? Well, take eg. the metric $\underset{\sim}{g}$,

$$g'_{\alpha\beta} = g(\underset{\sim}{e}'_\alpha, \underset{\sim}{e}'_\beta)$$

$$= g\left(\frac{\partial}{\partial x'^\alpha}, \frac{\partial}{\partial x'^\beta}\right)$$

$$= g\left(\frac{\partial x^\gamma}{\partial x'^\alpha}\frac{\partial}{\partial x^\gamma}, \frac{\partial x^\delta}{\partial x'^\beta}\frac{\partial}{\partial x^\delta}\right)$$

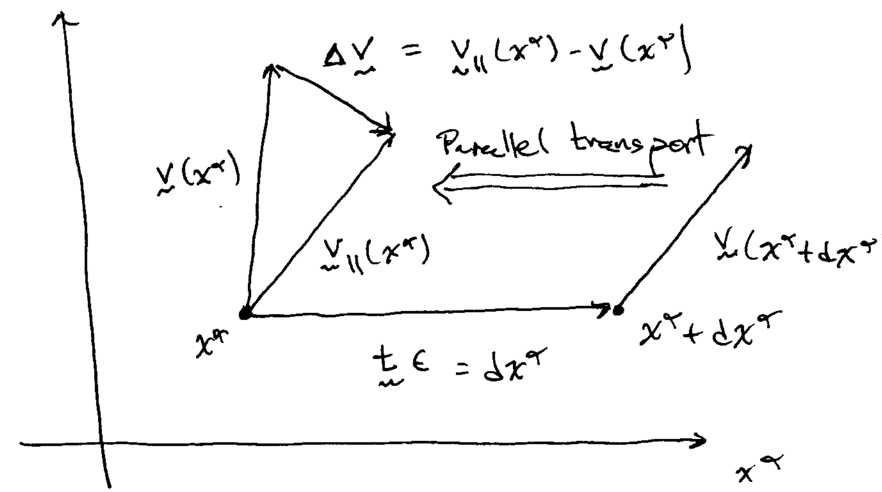
$$= \frac{\partial x^\gamma}{\partial x'^\alpha}\frac{\partial x^\delta}{\partial x'^\beta} g\left(\frac{\partial}{\partial x^\gamma}, \frac{\partial}{\partial x^\delta}\right)$$

$$= \frac{\partial x^\gamma}{\partial x'^\alpha}\frac{\partial x^\delta}{\partial x'^\beta} g_{\gamma\delta}$$ ✓ Same as before!

In general each index gets an appropriate factor of $\frac{\partial x}{\partial x'}$ or $\frac{\partial x'}{\partial x}$.

III Covariant Derivative

We would like to define a derivative of vector fields but we know that vectors based at different pts. live in different tangent spaces.



A reasonable strategy in flat space is depicted at right; the main ingredient is a notion of parallel transport. If we have such a notion then we can define:

$$\nabla_{\underset{\sim}{t}} \underset{\sim}{V}(x^\alpha) \equiv \lim_{\epsilon \to 0} \frac{[\underset{\sim}{V}(x^\alpha + t^\alpha \epsilon)]_{\parallel \text{ trans to } x^\alpha} - \underset{\sim}{V}(x^\alpha)}{\epsilon}$$

In rectangular coords of flat space or in a Local Inertial Frame (LIF) the components ~~of~~ V^α don't change under $\parallel$ transport and so V^α is just like a multivariable function,

$$(\nabla_{\underset{\sim}{t}} \underset{\sim}{V})^\alpha = t^\beta \frac{\partial V^\alpha}{\partial x^\beta} \quad (\text{LIF})$$

remind you only holds in this frame.

(Aside: Often it is convenient to drop the $\underset{\sim}{t}$ and just use the coordinate direction, i.e. $\underset{\sim}{e}_\beta$, so that

$$(\nabla_{\underset{\sim}{e}_\beta} \underset{\sim}{V})^\alpha = (\underset{\sim}{e}_\beta)^\gamma \frac{\partial V^\alpha}{\partial x^\gamma}$$

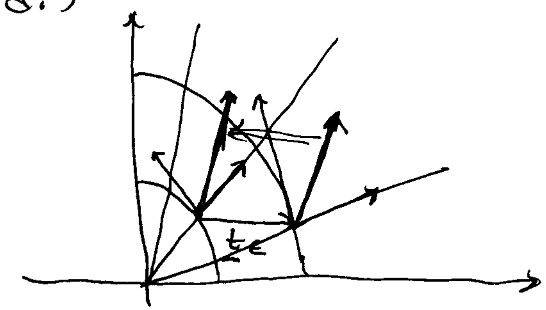
$$= \delta_\beta^{\ \gamma} \frac{\partial V^\gamma}{\partial x^\gamma} = \frac{\partial V^\alpha}{\partial x^\beta} \quad (\text{LIF})$$

This is just written,

$$\nabla_\beta V^\alpha = \frac{\partial V^\alpha}{\partial x^\beta} \quad (\text{LIF})$$

End Aside)

More generally, the components of $\underset{\sim}{v}$ do change under $\parallel$-transport — this is because the basis vectors change as you move around. For example in polar coord.s



for the derivative we get,

$$\nabla_\beta v^\alpha = \frac{\partial v^\alpha}{\partial x^\beta} + \tilde{\Gamma}^{\alpha}_{\ \beta\gamma}\, v^\gamma$$

To use this we need to find,

$$\tilde{\Gamma}^{\alpha}_{\ \beta\gamma} = ?$$

In addition to being extremal, geodesics are as straight as possible.

The key: For small $\epsilon \underset{\sim}{t}$ the change should be proportional to ϵt^α and to v^α.

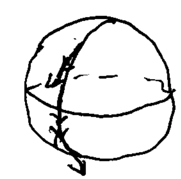
Then

$$v_{\parallel}^{\alpha}(x^\delta) = v^\alpha(x^\delta + \epsilon t^\delta) + \tilde{\Gamma}^{\alpha}_{\ \beta\gamma}(x^\delta)\, v^\gamma(x^\delta)\,(\epsilon t^\beta)$$

↑ proportionality

Putting this into our formula

meaning their tangent vectors parallel propagate into each other



This means that

$$\nabla_{\underset{\sim}{u}} \underset{\sim}{u} = 0 = u^\beta\left(\frac{\partial u^\alpha}{\partial x^\beta} + \tilde{\Gamma}^{\alpha}_{\ \beta\gamma}\, u^\gamma\right) = 0$$

TO BE CONTINUED ...