Today's Outline:

I Last Lecture

II Using Covariant Derivatives

III Curvature Begins

Lecture 24

April 17th, 2012

I Last Lecture

- Tensors are multilinear maps from a collection of vectors and dual vectors to a real number
- Raise & lower tensor indices with the metric
- Defined the covariant derivative:

0. Announcements

$$\nabla_{\underset{\sim}{t}} \underset{\sim}{V}(x^\alpha) = \lim_{\epsilon \to 0} \frac{[\underset{\sim}{V}(x^\alpha + \epsilon t^\alpha)]_{\parallel \text{ trans to } x^\alpha} - \underset{\sim}{V}(x^\alpha)}{\epsilon}$$

Using the key recognition that the change upon parallel transport should be proportional to v^α and to ϵt^α we found

$$\boxed{\nabla_\beta v^\alpha = \frac{\partial v^\alpha}{\partial x^\beta} + \tilde{\Gamma}^\alpha_{\beta\gamma} v^\gamma}$$

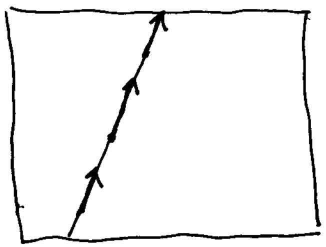
means $\underset{\sim}{t} = \underset{\sim}{e}_\beta$

II Using Covariant Derivatives

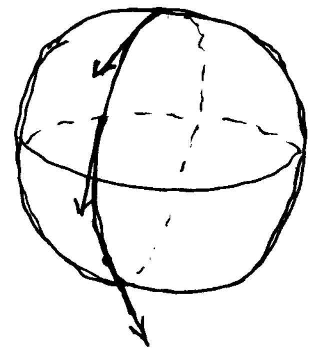
To use the previous formula we need to identify $\tilde{\Gamma}^\alpha_{\beta\gamma}$.

In addition to being extremal, geodesics are as straight as possible — meaning their tangent vectors parallel propogate into each other:

Pictorially,



or



This means that

$$(\nabla_u u)^\alpha = 0 = u^\beta\left(\frac{\partial u^\alpha}{\partial x^\beta} + \tilde{\Gamma}^\alpha_{\beta\gamma}\, u^\gamma\right)$$

and comparing this with the geodesic equation we find,

$$u^\beta\left(\frac{\partial u^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\beta\gamma}\, u^\gamma\right)$$

that $\tilde{\Gamma}^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma}$! This follows because at a point p there are geodesics in every possible direction u^γ and we can identify coefficients.

Let's do some calculations:

In special relativity we defined

$$a^\alpha = \frac{du^\alpha}{d\tau} \quad \text{(LIF only)}$$

but more generally we should define acceleration as something that takes you off of geodesics, i.e.,

$$a = \nabla_u u$$

This is sensible because it tells you how the 4-velocity changes as you move in the direction of the 4-velocity! Also in an LIF we find

$$(\nabla_u u)^\alpha = u^\beta\left(\frac{\partial u^\alpha}{\partial x^\beta} + \overset{0 \text{ bcse LIF}}{\Gamma^\alpha_{\beta\gamma}}\, u^\gamma\right)$$

$$= \frac{dx^\beta}{d\tau}\frac{\partial u^\alpha}{\partial x^\beta} = \frac{du^\alpha}{d\tau} \overset{\text{previous def.}}{=} a^\alpha \quad ✓$$

What acceleration is necessary to be stationary in Schwarzschild?

Well, stationary observer has,

$$(\underset{\sim}{u})^\alpha = (u^t, 0, 0, 0)$$

$$\Rightarrow \; -\left(1-\frac{2M}{r}\right)(u^t)^2 = -1 \;\Rightarrow\; u^t = \left(1-\frac{2M}{r}\right)^{-1/2}$$

and,

$$a^\alpha = u^\beta \nabla_\beta u^\alpha = u^t \nabla_t u^\alpha = u^t\left(\frac{\partial u^\alpha}{\partial t} + \Gamma^\alpha_{t\gamma} u^\gamma\right)$$

$$= u^t\left(\cancelto{0}{\frac{\partial u^\alpha}{\partial t}} + \Gamma^\alpha_{tt} u^t\right)$$

Using the Christoffel symbols we have

$$a^\alpha = (0, \Gamma^r_{tt} u^{t2}, 0, 0) = \left(0, \frac{M}{r^2}, 0, 0\right)$$

the other objects we've been thinking abt:

$$\nabla_\alpha f \equiv \frac{\partial f}{\partial x^\alpha}, \quad \nabla_{\underset{\sim}{u}} f \equiv u^\alpha \frac{\partial f}{\partial x^\alpha}$$

Now use the product rule (fancier to call it the Leibniz rule):

$$\nabla_\gamma v^\alpha w^\beta = v^\alpha \nabla_\gamma w^\beta + \nabla_\gamma (v^\alpha) w^\beta$$

But $v^\alpha w^\beta$ is a rank two tensor, so,

$$\nabla_\gamma t^{\alpha\beta} = \frac{\partial t^{\alpha\beta}}{\partial x^\gamma} + \Gamma^\alpha_{\gamma\delta} t^{\delta\beta} + \Gamma^\beta_{\gamma\delta} t^{\alpha\delta}$$

Interesting, it seems well behaved at $r=2M$, what's up?

Well,

$$(\underset{\sim}{a}\cdot\underset{\sim}{a})^{1/2} = \left(1-\frac{2M}{r}\right)^{-1/2}\frac{M}{r^2}$$

still diverges there.

<u>Covariant Deriv.s of everything else:</u>

Once you have a sensible definition for vectors, you can extend to

just like for $v^\alpha w^\beta$.

$$\nabla_\alpha v_\beta = \frac{\partial v_\beta}{\partial x^\alpha} - \Gamma^\gamma_{\alpha\beta} v_\gamma$$

and

$$\nabla_\gamma t^\alpha{}_\beta = \frac{\partial t^\alpha{}_\beta}{\partial x^\gamma} + \Gamma^\alpha_{\gamma\delta} t^\delta{}_\beta - \Gamma^\delta_{\gamma\beta} t^\alpha{}_\delta \text{ etc.}$$

<u>Long Example</u>: What are $\nabla_A v^B$? $\cancel{\nabla_\theta v^\phi}$, $\cancel{\nabla_\phi v^\theta}$

$$dS^2 = a^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Geodesics: $L = a^2\dot\theta^2 + \sin^2\theta\,\dot\phi^2 a^2 \quad \dot{} \equiv \frac{d}{dS}$

$$\frac{\partial L}{\partial \phi} = 0 \Rightarrow \frac{d}{dS}\left(\frac{\partial L}{\partial \dot\phi}\right) = \frac{d}{dS}\left(a^2\sin^2\theta\, 2\dot\phi\right) = 0$$

$$\Rightarrow \quad 2a^2\sin^2\theta\,\ddot{\phi} + 2a^2(2\sin\theta\cos\theta\,\dot{\theta})\dot{\phi} = 0$$

$$\Rightarrow \quad \ddot{\phi} + 2\cot\theta\,\dot{\theta}\dot{\phi} = 0$$

$$\Rightarrow \quad \boxed{\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta}$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \frac{d}{ds}\left(2a^2\dot{\theta}\right) = \frac{\partial L}{\partial\theta} = 2a^2\sin\theta\cos\theta\,\dot{\phi}^2$$

$$\Rightarrow \quad \ddot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0$$

$$\Rightarrow \quad \boxed{\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta}$$

$$\nabla_\phi v^\phi = \frac{\partial v^\phi}{\partial\phi} + \Gamma^{\phi}_{\phi\theta} v^\theta$$

$$= \frac{\partial v^\phi}{\partial\phi} + \cot\theta\, v^\theta.$$

End long example.

The ∇_α notation is very succinct. What is the Gyroscope eqn. in this notation?

$$\boxed{\nabla_{\underset{\sim}{u}} \underset{\sim}{S} = 0}$$

This says that the spin of a gyroscope is parallel transported along the gyro's geodesic!

All other Γ's are zero. Then,

$$\nabla_A v^B = \frac{\partial v^B}{\partial x^A} + \Gamma^B_{AC} v^C$$

$$\Rightarrow \nabla_\theta v^\theta = \frac{\partial v^\theta}{\partial\theta} + \cancel{\Gamma^\theta_{\theta C}}^{\,0} v^C = \frac{\partial v^\theta}{\partial\theta}$$

$$\nabla_\phi v^\theta = \frac{\partial v^\theta}{\partial\phi} + \Gamma^\theta_{\phi\phi} v^\phi$$

$$= \frac{\partial v^\theta}{\partial\phi} - \sin\theta\cos\theta\, v^\phi$$

$$\nabla_\theta v^\phi = \frac{\partial v^\phi}{\partial\theta} + \Gamma^\phi_{\theta\phi} v^\phi = \frac{\partial v^\phi}{\partial\theta} + \cot\theta\, v^\phi$$

III Curvature Begins

On the very first day I argued that Einstein's eqn. took the form

$$\boxed{???} = (G)\, T^{\mu\nu}$$

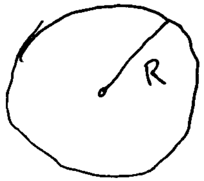
Some measure of curvature of Spacetime

source = energy momentum stress tensor

We need a theory of curvature ...

Theory of curvature:

(1) Circle:

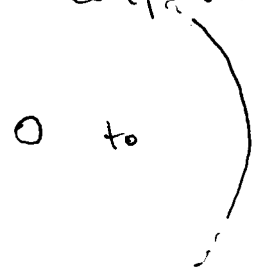


What is the curvature of this circle?

k (the "curvature") is

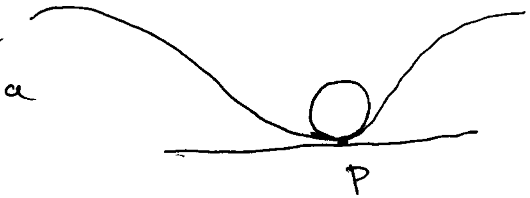
$$\boxed{k = \frac{1}{R}}$$

compare



(2) A curve

How to define curvature of a curve at the point P?

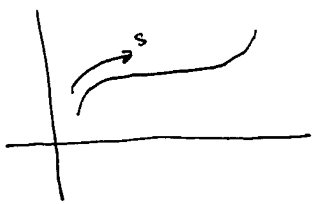


Draw the osculating (best fit) circle at P.

$$\boxed{k = \frac{1}{R|_p}}$$

and as a vector it points towards the center of the osculating circle.

(3) How to calculate $\vec{k}$



As usual let s be distance along a curve and let $\vec{r}(s)$ define the curve.

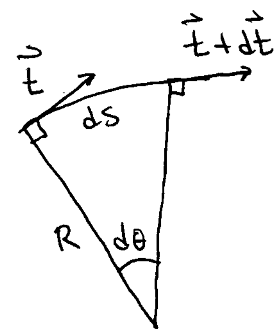
Then, $\vec{t} = \frac{d\vec{r}}{ds}$ is the unit tangent vector

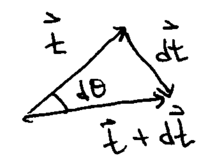
and we can define

$$\vec{k} = \frac{d\vec{t}}{ds} = \frac{d^2\vec{r}}{ds^2}$$

as the curvature.

This is sensible,





$d\vec{t}$ points toward the center of the circle $\Rightarrow \hat{k} = \hat{dt}$ and

$$\frac{dt}{t} = d\theta = \frac{ds}{R} \Rightarrow \frac{dt}{ds} = \frac{1}{R} = |\vec{k}|.$$

Work this out for a circle!