Lecture 25

April 19th, 2012

Today's Outline

I Last Lecture

II Curvature in general

I Last Lecture

- Interpretted the geodesic equation as

$$\boxed{\nabla_{\underset{\sim}{u}} \underset{\sim}{u} = 0} \left(= \underset{\sim}{a} \right)$$

which says the tangent to a geodesic is parallel transported into itself along the geodesic.

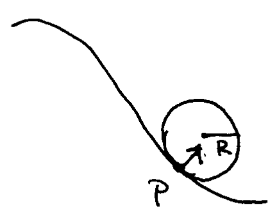
- Extended the covariant derivative to a general tensor,

$$\nabla_\gamma t^\alpha_{\ \beta} = \frac{\partial t^\alpha_{\ \beta}}{\partial x^\gamma} + \Gamma^\alpha_{\gamma\delta} t^\delta_{\ \beta} - \Gamma^\delta_{\gamma\beta} t^\alpha_{\ \delta}$$

- Introduced curvature of curves,

$$k = \frac{1}{R} \hat{k}$$

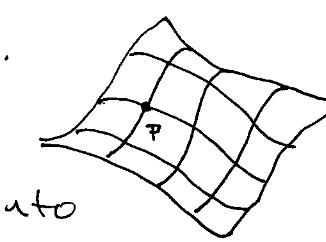
(points toward center)



II Curvature in general

Curvature of a surface

Strategy: draw a line (in the surface) through P.



Its curvature vector can be resolved into a component $\perp$ to the surface (the "normal" curvature) and a component $\parallel$ to the surf. ("geodesic" curvtre)

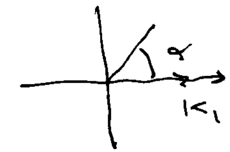
$$\boxed{\vec{k} = \underbrace{K\,\hat{n}}_{\text{Normal}} + \underbrace{\vec{k}_g}_{\text{geodesic}}}$$

The normal curv. is the same for all curves passing through P in the same direction – $\vec{k}_g$ is different.

Of course, the normal curv. does depend on the direction through p. Let K_1 be the maximum normal curv., and K_2 be the minimum:

Let α be the angle away from the max direction.

Euler's Formula:



$$\boxed{K(\alpha) = K_1 \cos^2\alpha + K_2 \sin^2\alpha}$$

(the max & min are at 90°)

K_1 and K_2 called the principal normal curvatures.

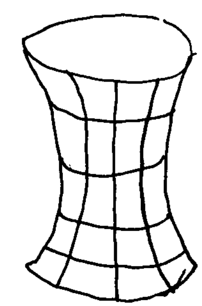
Ambiguity: $\hat{n}$ could be "up" or "down"

Convention: draw $\hat{n}$ such that K_1 (the max normal curv.) is positive, then K_2 could be pos., neg. or zero.

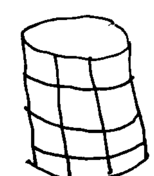
Classification:

(1) If K_2 is pos., p is called an elliptic pt. (e.g. any point on an ellipsoid)

(2) If K_2 is negative p is called a hyperbolic pt



(3) If one (or both) is zero p is a parabolic pt. (e.g. any pt in a plane or any pt on a cylinder)



If $K_1 = K_2$ the pt. p is called a navel.

Examples: (1) cylinder: $K_1 = \frac{1}{R}$, $K_2 = 0$.

(2) Sphere: $K_1 = K_2 = \frac{1}{R}$

(3) A general ellipsoid has four navels. (can have more, but in general there are 4).

Extrinsic / Intrinsic Curvature

Looking at a surface from point of view of imbedding in 3 space.

Intrinsic: From point of view of someone constrained to work in surface - can measure distance, angles, but no access to 3rd dim.

Intrinsic geom. unchanged by "rolling" - "bending" (as opposed to stretching or tearing).

Gauss & Bending Invariants

Note that the principal normal curvatures are not bending invariants. But their product, $K_1 K_2$, is a bending invariant, call it K. (Gaussian Curvature) Remarkable theorem whose proof you should check out.

This leads to a remarkable idea - can we ~~compute~~ characterize curvature completely using bending invariants (i.e. intrinsically).

Riemann's answer: Yes!

In n dimensions there are $\frac{1}{12} n^2 (n^2-1)$ curvatures that can be collected into a tensor.

Table

dim n	type	intrinsic curvatures
1	Line	0
2	Surface	$\frac{1}{12} 4 \cdot 3 = 1$
3	Space	$\frac{1}{12} 9 \cdot 8 = 6$
4	Space time	$\frac{1}{12} 16 \cdot 15 = 20$

captured by Riemann curvature

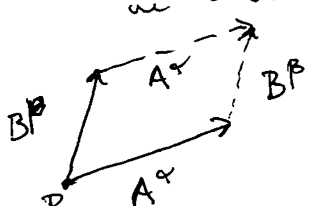
Riemann Tensor

How do we get at these invariants? There are (atleast) two ways:

(1) In flat space geodesics that are initially $\parallel$ remain so, this changes in curved spaces

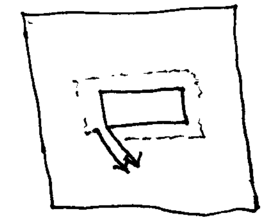
Hartle does (1), we'll do (2).

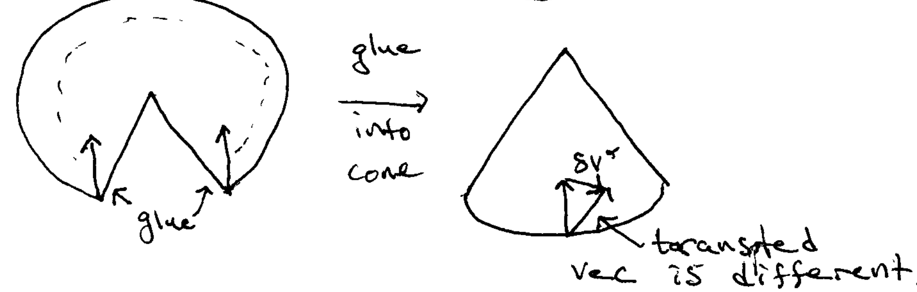
More precisely let's $\parallel$-transport around a parallelogram whose edges are $\underset{\sim}{A}$ and $\underset{\sim}{B}$



If we carry a vector $\underset{\sim}{V}$ around the loop its change upon returning to the initial pt is

(2) In flat space if you $\parallel$-transport a vector around a loop it is unaffected. Not true in a curved space



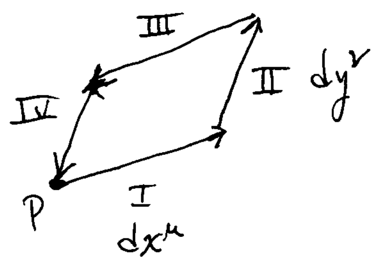


$$V^\alpha_{\substack{\parallel\text{-trans.}\\ \text{around loop}}}(p) - V^\alpha(p) \equiv \delta V^\alpha(p)$$

We want to characterize the curvature at the pt. p and so the vectors $\underset{\sim}{A}$ and $\underset{\sim}{B}$ should be small, so let $A^\alpha = dx^\alpha$ and $B^\beta = dy^\beta$ then the Riemann tensor is a tensor that takes V^β, dx^γ, dy^δ as input and returns δV^α:

$$\boxed{\delta V^\alpha = - R^\alpha{}_{\beta\gamma\delta}\, V^\beta dx^\gamma dy^\delta}$$

To find R in terms of the Γ's we divide our path into 4 pieces



<u>Step I</u>: $\delta V^\alpha_{I} = V^\alpha_{\parallel}(p+dx^\gamma) - V^\alpha(p)$

$$= -\Gamma^\alpha_{\beta\gamma} V^\beta dx^\gamma$$

[Aside about minus sign above:

When we introduced $\parallel$-transport we discussed backwards transport (see Fig 20.2 of Hartle), and found

$$V^\alpha_\parallel(x^\gamma) = V^\alpha(x^\gamma + dx^\gamma) + \Gamma^\alpha_{\beta\gamma} V^\beta dx^\gamma.$$

If we negate dx^γ,

$$V^\alpha_\parallel(x^\gamma) = V^\alpha(x^\gamma - dx^\gamma) - \Gamma^\alpha_{\beta\gamma} V^\beta dx^\gamma$$

and evaluate at $x^\gamma + dx^\gamma$ we find

$$V^\alpha_\parallel(x^\gamma + dx^\gamma) = V^\alpha(x^\gamma) - \Gamma^\alpha_{\beta\gamma}(x^\gamma + dx^\gamma) V^\beta(x^\gamma + dx^\gamma) dx^\gamma$$

Now, Taylor expand Γ and V in the second term a drop all higher order terms in dx^γ to find,

$$V^\alpha_\parallel(x^\gamma + dx^\gamma) = V^\alpha(x^\gamma) - \Gamma^\alpha_{\beta\gamma}(x^\gamma) V^\beta(x^\gamma) dx^\gamma$$

$$\Rightarrow \delta V^\alpha = V^\alpha_\parallel(p+dx^\gamma) - V^\alpha(p) = -\Gamma^\alpha_{\beta\gamma} V^\beta dx^\gamma$$

as claimed above. End Aside.]

<u>Step II</u>: $\delta V^\alpha_{II} =$ "$-\Gamma^\alpha_{\beta\gamma} V^\beta dy^\gamma$"

really $\rightarrow$

$$= -\Gamma^\alpha_{\beta\gamma}(p+dx) V^\beta(p+dx) dy^\gamma$$

Taylor expand Γ $\partial/\partial x^\mu$

transport V

$$\doteq -\left(\Gamma^\alpha_{\beta\gamma} + \partial_\mu \Gamma^\alpha_{\beta\gamma} dx^\mu\right)\left(V^\beta - \Gamma^\beta_{\sigma\tau} V^\sigma dx^\tau\right) dy^\gamma$$

$$= -\Gamma^\alpha_{\beta\gamma} V^\beta dy^\gamma + \Gamma^\alpha_{\beta\gamma}\Gamma^\beta_{\sigma\tau} V^\sigma dx^\tau dy^\gamma - (\partial_\mu \Gamma^\alpha_{\beta\gamma}) dx^\mu V^\beta dy^\gamma + O(dx^3)$$

<u>Step IV</u>: $\delta V^\alpha_{IV} = -\Gamma^\alpha_{\beta\gamma} V^\beta (-dy^\gamma)$

$$= \Gamma^\alpha_{\beta\gamma} V^\beta dy^\gamma$$

Step III: $\delta V^\alpha_{III} =$ "$-\Gamma^\alpha_{\beta\gamma} V^\beta (-dx^\gamma)$"

really

$$= \Gamma^\alpha_{\beta\gamma}(p+dy)\, V^\beta(p+dy)\, dx^\gamma$$

Taylor expand Γ
Transport V

$$\approx \left(\Gamma^\alpha_{\beta\gamma} + (\partial_\mu \Gamma^\alpha_{\beta\gamma})\, dy^\mu\right)\left(V^\beta - \Gamma^\beta_{\sigma\tau} V^\sigma dy^\tau\right) dx^\gamma$$

$$= \Gamma^\alpha_{\beta\gamma} V^\beta dx^\gamma - \Gamma^\alpha_{\beta\gamma}\Gamma^\beta_{\sigma\tau} V^\sigma dy^\tau dx^\gamma + (\partial_\mu \Gamma^\alpha_{\beta\gamma})\, dy^\mu V^\beta dx^\gamma + O(dx^3)$$

Total $\delta V^\alpha_I + \delta V^\alpha_{II} + \delta V^\alpha_{III} + \delta V^\alpha_{IV}$

$$\delta V^\alpha = -\Gamma^\alpha_{\beta\gamma} V^\beta dx^\gamma - \Gamma^\alpha_{\beta\gamma} V^\beta dy^\gamma$$
$$+ \Gamma^\alpha_{\beta\gamma}\Gamma^\beta_{\sigma\tau} V^\sigma dx^\tau dy^\gamma - (\partial_\mu \Gamma^\alpha_{\beta\gamma})\, dx^\mu V^\beta dy^\gamma$$
$$+ \Gamma^\alpha_{\beta\gamma} V^\beta dy^\gamma + \Gamma^\alpha_{\beta\gamma} V^\beta dx^\gamma - \Gamma^\alpha_{\beta\gamma}\Gamma^\beta_{\sigma\tau} V^\sigma dy^\tau dx^\gamma$$
$$+ (\partial_\mu \Gamma^\alpha_{\beta\gamma})\, dy^\mu V^\beta dx^\gamma$$

$$\Rightarrow \delta V^\alpha = \left[-\partial_\mu \Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\rho\mu}\Gamma^\rho_{\beta\gamma} + \partial_\gamma \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\rho\gamma}\Gamma^\rho_{\beta\mu}\right] dx^\mu V^\beta dy^\gamma$$

$$\Rightarrow \boxed{R^\alpha{}_{\beta\mu\gamma} = \left[\partial_\mu \Gamma^\alpha_{\beta\gamma} + \Gamma^\alpha_{\rho\mu}\Gamma^\rho_{\beta\gamma}\right] - \left[\partial_\gamma \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\rho\gamma}\Gamma^\rho_{\beta\mu}\right]}$$

This is the formula for the Riemann tensor that we were seeking.