Today's Outline

Lecture 26

April 24th, 2012

I Last Lecture

II Is it possible to derive Einstein's Eqns?

III Meaning of the Einstein Eqns

IV Connecting to the tensor formulation of the Einstein Eqn.

I Last Lecture

- Introduced the Gaussian curvature for 2-surfaces:

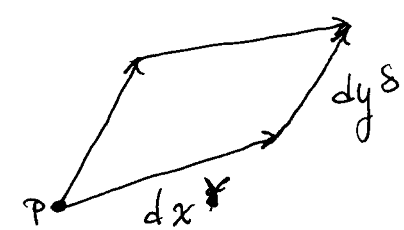
$$K = K_1 K_2$$

with K_1, K_2 the principal normal curvatures.

- K is a bending invariant.
- Discussed Riemann's remarkable derivation and summary of all bending invariants in a single tensor, the Riemann ~~tensor~~ (Curvature) tensor,

$$R^\alpha{}_{\beta\gamma\delta} = \frac{\partial \Gamma^\alpha_{\beta\delta}}{\partial x^\gamma} - \frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial x^\delta} + \Gamma^\alpha_{\gamma\epsilon}\Gamma^\epsilon_{\beta\delta} - \Gamma^\alpha_{\delta\epsilon}\Gamma^\epsilon_{\beta\gamma}$$

- This tensor takes in V^β, dx^γ and dy^δ and returns δV^α, the change in V^β upon parallel transport around the small parallelogram



that is,

$$\boxed{\delta V^\alpha = -R^\alpha{}_{\beta\gamma\delta} V^\beta dx^\gamma dy^\delta}$$

$$\delta V^\alpha \equiv V^\alpha_{\text{ll-trans around loop}}(p) - V^\alpha(p).$$

II Is it possible to derive Einstein's eqn.s?

Class discussion of this question.

If it's difficult to get the conversation started, try ~~the same~~ a similar question from a more familiar context:

Is it possible to derive Newton's laws for mechanics?

<u>Setup</u>: In S.R.: relative velocity, global inertial frame (coord.s), forces

In G.R.: relative velocity only at a pt. (same tangent space), at cost of some error can extend over small spacetime volume - local inertial frame (Riemann normal coord.s), geometry, test particles

III Meaning of the Einstein eqn.s

To understand S.R. and mechanics we begin in one frame and then generalize to all frames later. We'll do the same for Einstein's eqn.s (EE) here.

Title and approach adapted from Baez and Bunn.

<u>Einstein Equation</u>:

Consider a small ball of test particles initially at rest relative to each other (LIF). This ball deforms into an ellipsoid, at second order in time, as time passes.

An ellipsoid because any linear deformation of a ball is an ellipsoid and second order in time because the test particles start at relative rest

Let $V(t)$ be the volume of the ball, with t the proper time as measured by central test mass, then the Einstein equation is, $(G=c=1)$

$$\frac{\ddot{V}}{V}\Big|_{t=0} = -4\pi\left(\begin{array}{llll} \text{flow of} & t\text{-momentum} & \text{in} & t \text{ direction} \\ + \;'' & x\text{-mom} & '' & x \text{ dir.} \\ + \;'' & y\text{-mom} & '' & y \text{ dir.} \\ + \;'' & z\text{-mom} & '' & z \text{ dir.} \end{array}\right)$$

energ density, pressure in x-direction.

$$= -4\pi\left(\rho + P_x + P_y + P_z\right)$$

In order for this single equation to capture the full tensorial Einstein eq. we need to add that it should be valid for balls that begin at rest in all possible local inertial reference frames.

The generic arisal of ellipsoids ~~explains why gravity is~~ and the tidal nature of gravity are pleasantly consistent with this equation:

The R.H.S. consists of the diagonal elements of the "Stress-energy tensor" $T_{\alpha\beta}$. Notice that energy (mass) and pressure ~~tend to make~~ cause the ball of test particles to contract! This is gravity.

The flows are measured at the center of the ball at $t=0$ using the LIF.

Consider a ball of coffee grounds near the surface of the earth — the tidal nature of gravity stretches the ball into an ellipsoid and our equation says ($\rho = P_i = 0$ outside earth),

$$\ddot{V} = 0$$

In order to stretch in one direction the ball had to contract in another to maintain the same vol

<u>Pressure</u>: We aren't accustomed to it but pressure also contributes to gravitational attraction. In units with G and c we have

dominated by rest energy usually

$$\left.\frac{\ddot{V}}{V}\right|_{t=0} = -4\pi G\left(\rho_m + \frac{1}{c^2}(P_x+P_y+P_z)\right)$$

small

makes P contrib. even smaller!

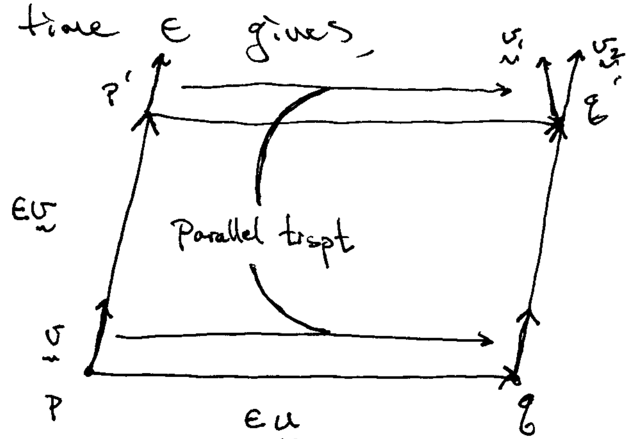
However, pressure term is significant, for example in stellar collapse of a neutron star.

IV Connecting to the tensor formulation of the Einstein eqn.

We make this connection in several steps.

<u>Step 1</u>: Riemann curvature to Geodesic deviation

We can use the Riemann tensor to calculate the acceleration of nearby particles in free fall. Assume initially they are at rest with respect to one another. Geodesic evolution for time ϵ gives,



The relative velocity at $t=\epsilon$ is,

$$\underset{\sim}{v_2} - \underset{\sim}{v_1}$$

Then the average acceleration is,

$$\underset{\sim}{a} = \frac{\underset{\sim}{v_2} - \underset{\sim}{v_1}}{\epsilon}.$$

But from last class,

$$\lim_{\epsilon \to 0} \frac{\underset{\sim}{v_2} - \underset{\sim}{v_1}}{\epsilon^2} = \underset{\sim}{R}(\underset{\sim}{u}, \underset{\sim}{v})\underset{\sim}{v}$$

So,

$$\lim_{\epsilon \to 0} \frac{\underset{\sim}{a}}{\epsilon} = \underset{\sim}{R}(\underset{\sim}{u}, \underset{\sim}{v})\underset{\sim}{v}$$

Using antisymmetry of the Riemann tensor, $\mathbf{R}(\mathbf{u},\mathbf{v})\mathbf{w} = -\mathbf{R}(\mathbf{v},\mathbf{u})\mathbf{v}$, and writing this in components gives,

$$\lim_{\epsilon\to 0} \frac{a^\alpha}{\epsilon} = - R^\alpha{}_{\beta\gamma\delta}\, v^\beta u^\gamma v^\delta$$

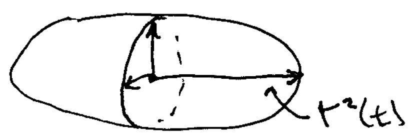
This is the geodesic deviation equation that Hartle also discusses.

<u>Step 2</u>: Acceleration to time evolution of volume

We choose a LIF centered on the ball and such that the center doesn't accelerate to 2^{nd} order in time. We know,

ball → ellipsoid,

choose the axes of the frame such that they align with the axes of the ellipsoid and let $r^j(t)$ be the radii of these axes,



If the initial ball radius is ϵ, we have

$$r^j(t) = \epsilon + \frac{1}{2} a^j t^2 + O(t^3)$$

and so,

$$\lim_{t\to 0} \frac{\ddot{r}^j}{r^j} = \lim_{t\to 0} \frac{a^j}{\epsilon}$$

For this setup $v^\alpha = (1,0,0,0)$ the velocity of the ball, and $\mathbf{u}$ takes each coord. axis value one at a time so that,

$$\lim_{t\to 0} \frac{\ddot{r}^j(t)}{r^j} = - R^j{}_{\beta j \delta}\, v^\beta v^\delta \quad \text{(no sum on } j)$$

$$= - R^j{}_{tjt} \quad \text{(no sum on } j)$$

Continued in Lecture 27 ...