Lecture 27

April 26th, 2012

Final Lecture!

Today's Outline

I Last Lecture

II Connecting to the tensor formulation of the Einstein equation

I Last Lecture

- Introduced the Einstein equation in the form:

$$\left.\frac{\ddot{V}}{V}\right|_{t=0} = -4\pi \begin{pmatrix} \text{flow of } t\text{-mom. in } t \text{ dir.} \\ + \text{ `` } \quad x\text{-mom in } x \text{ dir.} \\ + \text{ `` } \quad y\text{-mom in } y \text{ dir.} \\ + \text{ `` } \quad z\text{-mom in } z \text{ dir.} \end{pmatrix}$$

$$= -4\pi\left(\rho + P_x + P_y + P_z\right)$$

(units w/ $G = c = 1$).

To capture full Einstein eqn we require this to hold for balls of test particles ~~at~~ initially at rest in all inertial frames.

- We began the process of connecting this formulation with the full tensorial one:

Step 1: showed $\lim_{\epsilon \to 0} \frac{a^\alpha}{\epsilon} = -R^\alpha{}_{\beta\gamma\delta}\, v^\beta u^\gamma v^\delta$

the geodesic deviation equation.

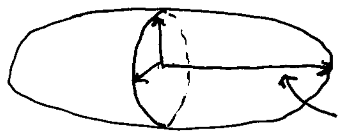
II Tensor Formulation

We continue with Step 2: Acceleration to time evolution of the volume.

Choose a LIF centered on the ball and such that the center doesn't accelerate to 2nd order in time. We know,

ball $\rightarrow$ ellipsoid,

so choose axes of the frame such that they align with the axes of the ellipsoid and let $r^j(t)$ $(j=1,2,3)$ be the radii of these axes,



$r^2(t)$

If the initial ball radius is ϵ, we have

$$r^j(t) = \epsilon + \tfrac{1}{2} a^j t^2 + O(t^3)$$

and so,

$$\lim_{t\to 0} \frac{\ddot{r}^j}{r^j} = \lim_{t\to 0} \frac{a^j}{\epsilon}$$

Because $V \propto r_1 r_2 r_3$ we have

$$\frac{\ddot{V}}{V} \xrightarrow[t\to 0]{} \frac{\ddot{r}^1}{r^1} + \frac{\ddot{r}^2}{r^2} + \frac{\ddot{r}^3}{r^3}$$

(the cross terms involving $\dot{r}^j$ fall out as $t \to 0$ because we assumed $\dot{r}^j(0)=0$).

So,

$$\underbrace{\lim_{V\to 0}}_{\text{condition for LIF}} \left.\frac{\ddot{V}}{V}\right|_{t=0} = -R^\alpha{}_{t\alpha t} \quad \left(\begin{matrix}\text{yes, sum}\\ \text{on } \alpha\end{matrix}\right)$$

The sum is over all values of α $(\alpha = 0,1,2,3)$

With this setup the initial 4-velocity of the ball is purely timelike $v^\alpha = (1,0,0,0)$ and and we can take $\underset{\sim}{u}$ to be each coord. axis one at a time, so that,

$$\lim_{t\to 0} \frac{\ddot{r}^j(t)}{r^j(t)} = -R^j{}_{\beta j \delta} v^\beta v^\delta \quad \left(\begin{matrix}\text{no}\\ \text{sum}\\ \text{on } j\end{matrix}\right)$$

$$= -R^j{}_{tjt} \quad \left(\begin{matrix}\text{no sum}\\ \text{on } j\end{matrix}\right)$$

because $R^t{}_{ttt} = 0$ and it's easier to write as a sum over all values.

When we contract over two indices of the Riemann tensor in this pattern we call the result the Ricci tensor:

$$R_{\alpha\beta} = R^\gamma{}_{\alpha\gamma\beta}$$

Ricci tensor

$$= R^0{}_{\alpha 0\beta} + R^1{}_{\alpha 1\beta} + R^2{}_{\alpha 2\beta} + R^3{}_{\alpha 3\beta}$$

So, we've found,

$$\frac{\ddot{V}}{V}\Big|_{t=0} = -R_{tt}$$

← time-time component of the Ricci tensor.

Our ball of test particles is responding to the curvature of spacetime!

Step 3: Matter-Energy-Pressure Causes Curvature

~~Einsteins~~ A facet of Einstein's insight was that Mass-Energy-Pressure causes this curvature. Simply stated, the ball accelerates due to gravitational attraction, that is,

$$\frac{\ddot{V}}{V}\Big|_{t=0} = -4\pi(\rho + P_x + P_y + P_z)$$

$$= -4\pi(T_{tt} + T_{xx} + T_{yy} + T_{zz})$$

Here I've reintroduced the stress-energy tensor $T_{\alpha\beta}$. As a tensor it eats two vectors – first one tells you which momentum and second in which direction.

Step 4: General Frame

Putting it all together we find

$$R_{tt} = 4\pi(T_{tt} + T_{xx} + T_{yy} + T_{zz})$$

We've been working in ~~an~~ LIF, which has metric ~~diag~~ $g_{\alpha\beta} = \eta_{\alpha\beta} = \text{diag}(-1,1,1,1)$

It would be convenient to write $T^\gamma{}_\gamma$ but

$$T^\gamma{}_\gamma = -T_{tt} + T_{xx} + T_{yy} + T_{zz}$$

← because of metric

But we can fix this with $g_{tt} = -1$,

$$T_{tt} - \tfrac{1}{2} g_{tt} T^\gamma{}_\gamma = \cancel{\tfrac{1}{2}(T_{tt} + \cdots + T_{zz})}$$

$$T_{tt} + \tfrac{1}{2}(-T_{tt} + \cdots + T_{zz})$$

$$= \tfrac{1}{2}(T_{tt} + \cdots + T_{zz})$$

Then we have

$$R_{tt} = 8\pi(T_{tt} - \tfrac{1}{2} g_{tt} T^\gamma{}_\gamma)$$

To promote this to any frame we require that it hold as a tensor equation:

$$R_{\alpha\beta} = 8\pi(T_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} T^\gamma{}_\gamma)$$

Our first version of Einstein's eqn!

Traditionally people prefer the matter side be simple, so note,

$$R^\beta{}_\beta = 8\pi\left(T^\beta{}_\beta - \tfrac{1}{2} g^\beta{}_\beta T^\gamma{}_\gamma\right)$$

$$= 8\pi\left(T^\beta{}_\beta - \tfrac{1}{2}\cdot 4\cdot T^\beta{}_\beta\right)$$

$$= -8\pi T^\beta{}_\beta$$

~~Putting this~~ We call $R^\beta{}_\beta \equiv R$ the Ricci scalar. Putting this back

$$\boxed{G_{\alpha\beta} = 8\pi T_{\alpha\beta}}$$

In units with G and c,

$$\boxed{G_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}}$$

A remarkable, perhaps the most remarkable, aspect of these eqns is that they are nonlinear.

into the E eqn we find,

$$R_{\alpha\beta} = 8\pi\left(T_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta}\left(-\tfrac{1}{8\pi} R\right)\right)$$

$$\Rightarrow R_{\alpha\beta} = 8\pi T_{\alpha\beta} + \tfrac{1}{2} g_{\alpha\beta} R$$

$$\Rightarrow \boxed{R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R = 8\pi T_{\alpha\beta}}$$

The Einstein Equation!

Abbreviations:

Einstein tensor $\rightarrow G_{\alpha\beta} \equiv R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R$

$$R^\alpha{}_{\beta\gamma\delta} = \frac{\partial \Gamma^\alpha_{\beta\delta}}{\partial x^\gamma} - \frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial x^\delta} + \Gamma^\alpha_{\gamma\epsilon}\Gamma^\epsilon_{\beta\delta} - \Gamma^\alpha_{\delta\epsilon}\Gamma^\epsilon_{\beta\gamma}$$

$$R^\gamma{}_{\alpha\gamma\beta} = R_{\alpha\beta} = \frac{\partial \Gamma^\gamma_{\alpha\beta}}{\partial x^\gamma} - \frac{\partial \Gamma^\gamma_{\alpha\gamma}}{\partial x^\beta} + \Gamma^\gamma_{\gamma\epsilon}\Gamma^\epsilon_{\alpha\beta} - \Gamma^\gamma_{\beta\epsilon}\Gamma^\epsilon_{\alpha\gamma} \quad \left(\text{sum on } \gamma\right)$$

$$\Gamma^\alpha_{\beta\gamma} = \tfrac{1}{2} g^{\alpha\delta}\left(\frac{\partial g_{\delta\beta}}{\partial x^\gamma} + \frac{\partial g_{\delta\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\delta}\right)$$

$G_{\alpha\beta} \overset{\text{contains}}{\supset}$ 2nd derivs of g, 1st derivs of g and many nonlinear terms