Today's Outline

Lecture 3

Jan. 24th, 2012

I. Geometry of Relativity

II. Recap relativistic effects & transformations

III Four Vectors

~~IV S.R. Kinematics & Dynamics~~

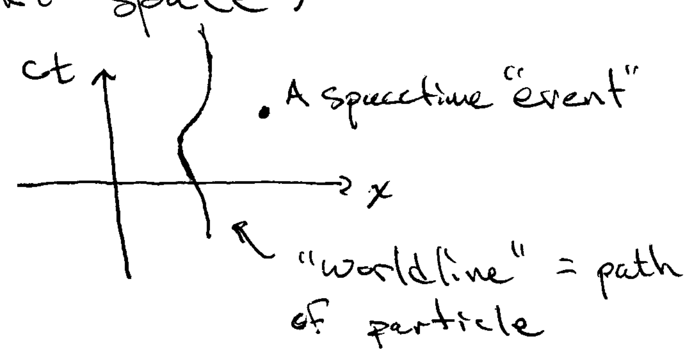
I Geometry of Relativity

Last week we decided we would study the geometry of space and time. We decided to do this locally using the line element

ds^2 = squared distance between nearby points

As you know, Einstein argued that we should study space and time together, that is spacetime.

~~We call this~~ Consider flat Spacetime (or Minkowski space)



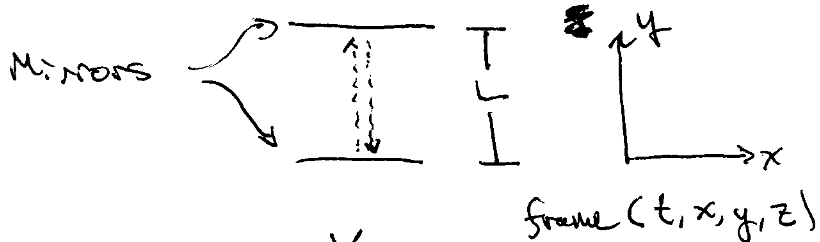
What is its geometry? i.e. ds^2?

Turn our earlier observation about coordinates around:

$\boxed{\text{Seek a distance all observers agree upon; an \underline{invariant} distance}}$

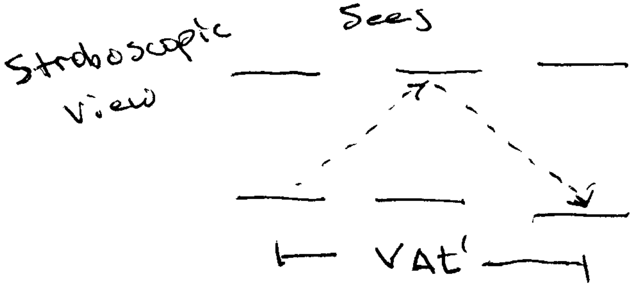
Invariant under constant speed (S.R.) changes of frame.

Motivating Example: A light clock



V

Moving frame (t', x', y', z') sees



In (t, x, y, z),

$$\Delta t = \frac{2L}{c} \qquad \Delta x = \Delta y = \Delta z = 0$$

while in (t', x', y', z') Check it!

$$\Delta t' = \frac{2}{c}\sqrt{L^2 + \left(\frac{\Delta x'}{2}\right)^2}, \quad \Delta x' = V\Delta t',$$

$$\Delta y' = \Delta z' = 0$$

These two frames agree on

$$-(c\Delta t')^2 + \Delta x'^2 + \Delta y'^2 + \Delta z'^2 = -4\left[L^2 + \frac{\Delta x'^2}{4}\right] + \Delta x'^2$$

$$= -4L^2 = -(c\Delta t)^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$$

Thus

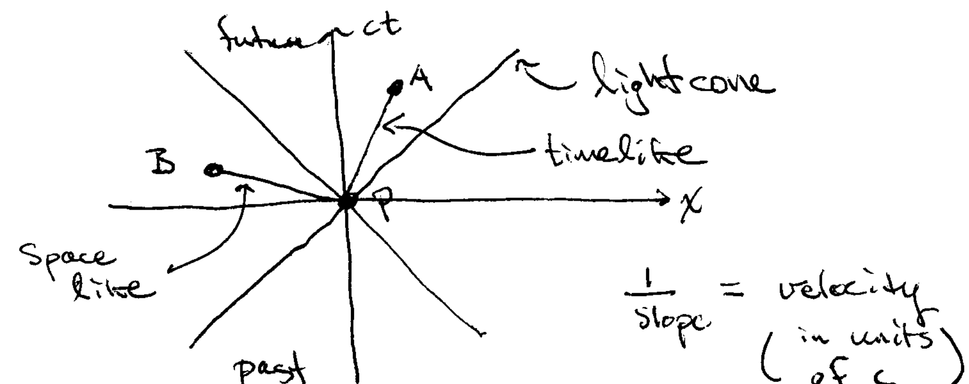
$$(\Delta S)^2 = -(c\Delta t)^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$$

= same primed

provides an invariant generalization of $dS^2 = dx^2 + dy^2 + dz^2$.

$$\boxed{ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2}$$

Flat spacetime line element

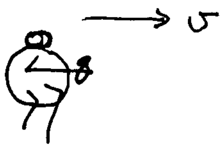


$(\Delta S)^2 > 0$ spacelike separated

$(\Delta S)^2 = 0$ null or lightlike "

$(\Delta S)^2 < 0$ timelike "

Proper Time:



particle

"proper time" on particle's own watch, τ

Wall Clock

Ordinary "coordinate-time", t

We will show briefly that

$$d\tau^2 = -ds^2/c^2$$

Thus for timelike curves = world lines proper time is a nice, Lorentz invariant, measure of "distance" along the curve.

II Recap Relativistic Effects

(i) Lorentz Contraction

Moving rods are contracted (shorter)

$$\underset{\text{length in moving frame}}{L} = \underset{\text{rest length}}{L_*}/\gamma \qquad \gamma \equiv \frac{1}{\sqrt{1-v^2/c^2}}$$

(ii) Time Dilation

Moving clocks run slow

$$dt = \gamma\, d\tau$$

(iii) Velocity Addition

$$V_{AC} = \frac{V_{AB} + V_{BC}}{1 + \frac{V_{AB}V_{BC}}{c^2}} \quad \left(\text{along boost}\right)$$

More specifically,

$$V^{x'} = \frac{V^x - v}{1 - vV^x/c^2} \qquad \gamma = \frac{1}{\sqrt{1-v^2/c^2}}$$

$$V^{y'} = \frac{V_y/\gamma}{1 - vV^x/c^2} \qquad V^{z'} = \frac{V_z/\gamma}{1 - vV^x/c^2}$$

All three effects (i), (ii), and (iii) follow from the relativity of simultaneity!

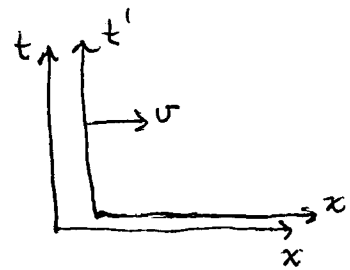
(See N.D. Mermin "It's about time")

Lorentz Transformations:

Here and in what follows we adopt units in which $c = 1$, see Hartle's excellent Appendix A.

The basic connection btwn frames is

$$t' = \gamma(t - v x)$$
$$x' = \gamma(x - v t)$$
$$y' = y$$
$$z' = z$$



Origins coincide at $t = t' = 0$

and view these as the components of a vector $\underset{\sim}{x}$ (I'll adopt Hartle's suggestion and use an under twiddle for 4-vectors).

4-vectors are vectors:

$$\alpha(\underset{\sim}{a} + \underset{\sim}{b}) = \alpha \underset{\sim}{a} + \alpha \underset{\sim}{b}$$

For calculating we need to express 4-vectors in a basis (recall

III Four Vectors

So far our notation doesn't reflect our recognition of the unification of space and time.

Lorentz transformation suggests we collect

$$x^\mu = (x^0, x^1, x^2, x^3)$$
$$\equiv (t, x, y, z)$$

$$\vec{x} = x\hat{x} + y\hat{y} + z\hat{z}\,)$$

$$\underset{\sim}{a} = a^\mu \underset{\sim}{e}_\mu$$
$$= a^t \underset{\sim}{e}_t + a^x \underset{\sim}{e}_x + a^y \underset{\sim}{e}_y + a^z \underset{\sim}{e}_z$$

Here we adopt the Einstein summation convention: repeated indices — one up, one down — are summed.

We also write

$$\underset{\sim}{a} = (a^t, a^i), \quad \underset{\sim}{a} = (a^t, \vec{a})$$

<u>Scalar Product</u>: Scalar product of basis vectors determines all scalar products, because,

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = (a^\mu \underset{\sim}{e}_\mu) \cdot (b^\nu \underset{\sim}{e}_\nu)$$
$$= a^\mu b^\nu \underset{\sim}{e}_\mu \cdot \underset{\sim}{e}_\nu$$
$$\equiv a^\mu b^\nu \eta_{\mu\nu}$$

$$ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu$$

and thought of this way, we call $\eta_{\mu\nu}$ the metric of flat spacetime (or Minkowski metric).

<u>Example</u>:

$$ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu$$
$$= -dt^2 + dx^2 + dy^2 + dz^2$$
$$= -dt^2 + v^2 dt^2$$

$$\eta_{\mu\nu} \equiv \underset{\sim}{e}_\mu \cdot \underset{\sim}{e}_\nu$$

To find $\eta_{\mu\nu}$ we require

$$(\Delta s)^2 = \Delta \underset{\sim}{x} \cdot \Delta \underset{\sim}{x}$$

but then

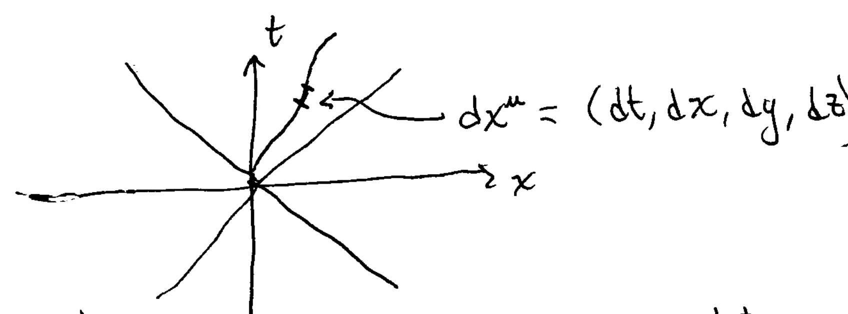
$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Clearly then

$$= -(1 - v^2)\, dt^2$$
$$= -\frac{dt^2}{\gamma^2} = -d\tau^2$$

as mentioned before

$$d\tau^2 = -ds^2 \; \left(\overset{\text{old units}}{=} -ds^2/c^2\right)$$



In general: $\underset{\sim}{a} \cdot \underset{\sim}{b} = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3 = -a^t b^t + \vec{a} \cdot \vec{b}$