Today's Outline

Lecture 4

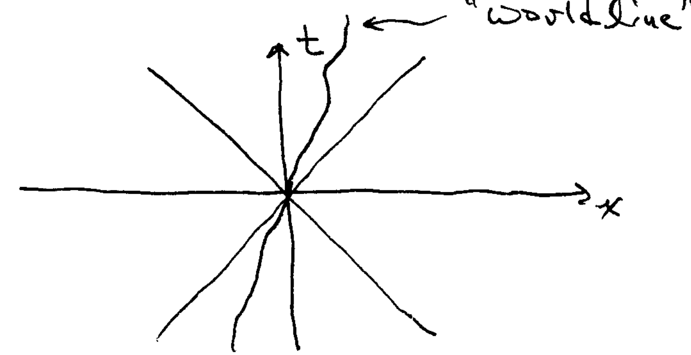
Jan 26th, 2012

I S.R. Kinematics & Dynamics

II Light

I S.R. Kinematics & Dynamics

Real particles follow timelike trajectories called worldlines



Kinematics:

Parametrize a worldline

$$x^\alpha = x^\alpha(\tau) = (t(\tau), x(\tau), y(\tau), z(\tau))$$

e.g. by proper time. Then,

Proper Velocity:

$$u^\alpha = \frac{dx^\alpha}{d\tau}$$

why choose proper time?

$$u^t = \frac{dt}{d\tau} = \gamma = \frac{1}{\sqrt{1-\vec{v}^2}}$$

$$u^x = \frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} = \gamma v^x = \frac{v^x}{\sqrt{1-\vec{v}^2}}$$

etc.

So,

$$\boxed{u^\alpha = (\gamma, \gamma\vec{v})}$$

Important calculation,

$$\underset{\sim}{u} \cdot \underset{\sim}{u} = \eta_{\alpha\beta} u^\alpha u^\beta = -\gamma^2 + \gamma^2 \vec{v}^2$$

$$= -\frac{(1-\vec{v}^2)}{(1-\vec{v}^2)} = -1.$$

Proper Acceleration: (Newton: $a = \frac{d^2x}{dt^2} = \frac{dv}{dt}$)

$$\boxed{a^\alpha = \frac{du^\alpha}{d\tau}}$$

Another important calculation:

$$\frac{d}{d\tau}(\underset{\sim}{u}\cdot\underset{\sim}{u}) = \frac{d}{d\tau}(\eta_{\alpha\beta}u^\alpha u^\beta)$$

$$= \eta_{\alpha\beta}a^\alpha u^\beta + \eta_{\alpha\beta}u^\alpha a^\beta$$

$$= \underset{\sim}{a}\cdot\underset{\sim}{u} + \underset{\sim}{u}\cdot\underset{\sim}{a} = 2\underset{\sim}{a}\cdot\underset{\sim}{u}$$

but also,

$$\frac{d}{d\tau}(\underset{\sim}{u}\cdot\underset{\sim}{u}) = \frac{d}{d\tau}(-1) = 0$$

$$\Rightarrow \boxed{\underset{\sim}{a}\cdot\underset{\sim}{u} = 0}\ !$$

(four)-acceleration is perp. to (four-) velocity!

Energy & Momentum:

Natural guess is correct:

$$\boxed{p^\alpha = m u^\alpha}$$

Energy-momentum four-vector

$$p^\alpha = (E, \vec{p})$$

Relativistic Energy:

$$E = mu^0 = m\gamma = \frac{m}{\sqrt{1-\vec{v}^2}}$$

$$\approx m\left(1+\frac{1}{2}\vec{v}^2+\dots\right)$$

$$= m + \frac{1}{2}m\vec{v}^2 + \dots$$

Note Also:

$$\frac{\vec{p}}{E} = \frac{\gamma m\vec{v}}{\gamma m} = \vec{v}$$

useful!

Relativistic Momentum: $\vec{p} = \gamma m\vec{v} = \frac{m\vec{v}}{\sqrt{1-\vec{v}^2}}$

Important calculation:

$$\underset{\sim}{p}^2 = \underset{\sim}{p}\cdot\underset{\sim}{p} = m^2\underset{\sim}{u}\cdot\underset{\sim}{u} = -m^2$$

But then,

$$\underset{\sim}{p}\cdot\underset{\sim}{p} = \eta_{\alpha\beta}p^\alpha p^\beta = -E^2 + \vec{p}^2$$

$$\Rightarrow \boxed{E = (m^2+p^2)^{1/2}}$$

Dynamics:

Newton's First law carries over in the form,

$$\frac{d\underset{\sim}{u}}{d\tau} = 0$$

$\Rightarrow$ straight line motion when there are no forces and $\vec{V} = \text{const}$.

Newton's 2nd law:

$$\underset{\sim}{f} = m\frac{d\underset{\sim}{u}}{d\tau} = m\underset{\sim}{a} = \frac{d\underset{\sim}{p}}{d\tau}$$

it is useful to also introduce the three-force

$$\vec{F} = \frac{d\vec{p}}{dt}$$

$\leftarrow$ relativistic momentum

$\leftarrow$ coordinate time

In terms of three-force we have

$$\vec{f} = \frac{d\vec{p}}{d\tau} = \frac{d\vec{p}}{dt}\frac{dt}{d\tau} = \gamma\vec{F}$$

Also, $\underset{\sim}{f}\cdot\underset{\sim}{u} = 0 \Rightarrow -f^t\gamma + \gamma^2\vec{F}\cdot\vec{V} = 0$

$$\Rightarrow f^t = \gamma\vec{F}\cdot\vec{V}$$

and we call $\underset{\sim}{f}$ the four-force. (or Minkowski force).

Note that,

$$\underset{\sim}{f}\cdot\underset{\sim}{u} = m\underset{\sim}{a}\cdot\underset{\sim}{u} = 0$$

So only three of the equations $\underset{\sim}{f} = m\underset{\sim}{a}$ are independent.

Despite its hybrid character

so,

$$\boxed{\underset{\sim}{f} = (\gamma\vec{F}\cdot\vec{V}, \gamma\vec{F})}$$

Note that

$$\frac{dp^0}{d\tau} = \frac{dp^0}{dt}\frac{dt}{d\tau} = \gamma\frac{dE}{dt} = f^0 = \gamma\vec{F}\cdot\vec{V}$$

$$\Rightarrow \frac{dE}{dt} = \vec{F}\cdot\vec{V}$$

Relativistic version of power — which we know follows from other E.O.M.

<u>Example</u>: Find $x(t)$ for motion in 1 dimension under a constant "three"-force F.

Say it starts at $x=0$, from rest at $t=0$. Well,

$$\vec{F} = \frac{d\vec{p}}{dt}$$

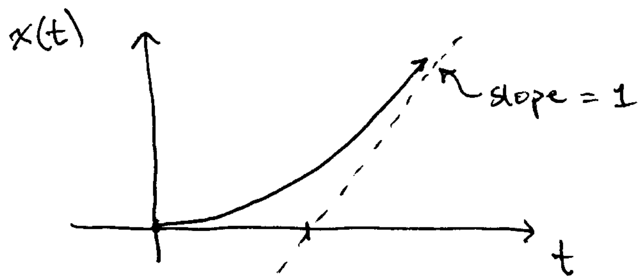
$$\Rightarrow dp = F\,dt \Rightarrow p = Ft$$

$$\Rightarrow \gamma m v = Ft \Rightarrow \frac{mV}{\sqrt{1-V^2}} = Ft$$

Separate and integrate again,

$$\int_0^x dx' = \int_0^t \frac{Ft'}{\sqrt{m^2+F^2t'^2}}\,dt'$$

$$\Rightarrow \boxed{x(t) = \frac{1}{F}\sqrt{m^2+F^2t^2} - \frac{m}{F}.}$$



Solve for V,

$$\frac{m^2V^2}{1-V^2} = F^2t^2$$

$$\Rightarrow m^2V^2 = F^2t^2 - F^2t^2V^2$$

$$\Rightarrow V^2(m^2+F^2t^2) = F^2t^2$$

$$\Rightarrow V = \frac{Ft}{\sqrt{m^2+F^2t^2}}$$

Plot

II Light

Light has the amazing property

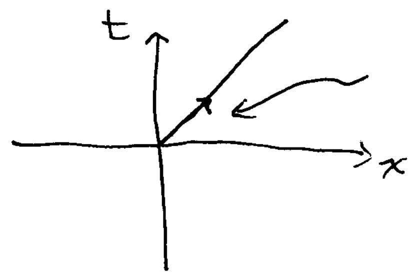
$$m = 0$$

and travels at $V=1$. We've seen that

$$ds^2 = -d\tau^2 = 0$$

for null separations. This means we can't use s or τ to parametrize light trajectories.

Instead invent a parametrization – e.g. for $x=t$ we take



i.e.

$$x^\alpha = \tilde{u}^\alpha \lambda, \quad \text{with } \tilde{u}^\alpha = (1,1,0,0)$$

Note that

$$u^\alpha = \frac{dx^\alpha}{d\lambda} = \tilde{u}^\alpha$$

and

$$\underset{\sim}{u} \cdot \underset{\sim}{u} = -1+1+0+0=0,$$

i.e. $\underset{\sim}{u}$ is a null vector.

Also,

$$\frac{d^2x^\alpha}{d\lambda^2} = \frac{du^\alpha}{d\lambda} = 0$$

just like a free particle.

If instead $x^\alpha = \tilde{u}^\alpha \sigma^3$, then

Same trajectory,

$$u^\alpha = \frac{dx^\alpha}{d\sigma} = (3\sigma^2, 3\sigma^2, 0, 0)$$

and $\underset{\sim}{u} \cdot \underset{\sim}{u} = 0$ still but

$$\frac{d^2x^\alpha}{d\sigma^2} = \frac{du^\alpha}{d\sigma} = 6\sigma\tilde{u}^\alpha \neq 0.$$

We call a parametrization of a light ray <u>affine</u> if it gives the free particle eq. for light rays.

Affine parameters are preffered, but not unique.

Energy & Momentum

Einstein recognized that for a photon,

$$E = \hbar\omega$$

Planck's const./2π ... freq.

The Energy-momentum relation

$$E^2 = m^2 + \vec{p}^{\,2}$$

continues to hold for light w/ $m=0$,

$$\Rightarrow |\vec{p}| = E$$

But then,

$$\vec{p} = \hbar \vec{k} \quad \text{w/} \quad |\vec{k}| = \omega$$

and $\hat{k}$ in direction of motion. So,

$$\boxed{\text{light:} \quad p^\alpha = (E, \vec{p}) = \hbar(\omega, \vec{k}) \equiv \hbar k^\alpha}$$

So,

$$\omega = \gamma(\omega' - V k^{x'})$$

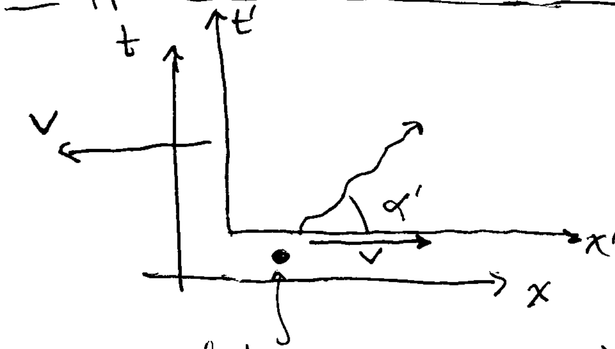
But $k^{x'} = \omega' \cos\alpha'$, so

$$\omega = \gamma(\omega' - V\omega'\cos\alpha')$$

$$\Rightarrow \boxed{\omega' = \omega \frac{\sqrt{1-V^2}}{1 - V\cos\alpha'}} \quad \text{Doppler Effect}$$

k^α is the wave four-vector.

Doppler Shift ~~and Rel. Beaming~~



light source moving in (t', x')

k^α is a 4-vector, transforms under Lorentz trans. just like x^μ