Today's Outline:

Lecture 6

Feb 2^{nd}, 2012

I Last Lecture

II Grav. Time Dilation as Geometry

III Particle Motion in Spacetime

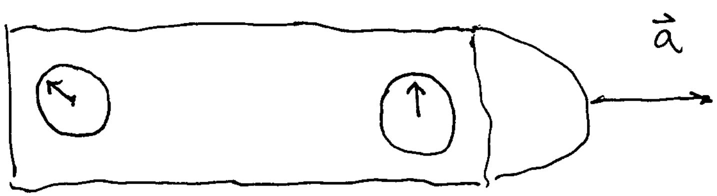
Comment on HW 2: Hartle 5.7

I Last Lecture

What is special about gravity? One answer, $m_I = m_G \Rightarrow$ the equivalence principle.

We know moving clocks run slow, what becomes of slouching ~~clocks?~~ or accelerated clocks?

clocks lower in a gravitational potential run slow.



How does this mesh with light falling down a mineshaft?

One way to explain the blueshift is to say the observer's clocks at the bottom run slow.

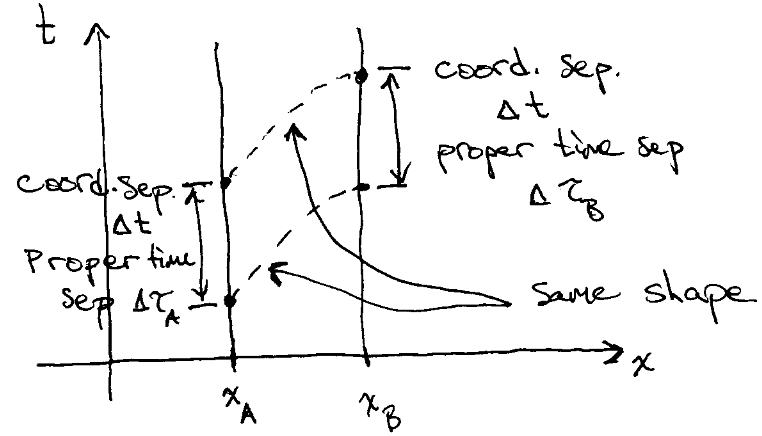
Static Weak field Metric:

$$ds^2 = -\left(1 + \frac{2\Phi(x^i)}{c^2}\right)(c\,dt)^2 + \left(1 - \frac{2\Phi(x^i)}{c^2}\right)(dx^2+dy^2+dz^2)$$

with Φ the gravitational potential.

II Grav. Time Dilation as Geometry

The static weak field metric describes a curved geometry. So, light rays will no longer be straight lines (we argued this from the equivalence princ. last time). But the metric <u>is</u> static, so the shape of the



trajectories of two light rays is the same.

Important calculation: what are $\Delta\tau_A$ and $\Delta\tau_B$ to first order in $\Phi(x^i)/c^2$?

$$\Delta\tau_A^2 = -\frac{\Delta S^2}{c^2} = \left(1 + \frac{2\Phi_A}{c^2}\right)\frac{c^2\Delta t^2}{c^2}$$

here $\Phi_A \equiv \Phi(x_A)$,

$$\Rightarrow \Delta\tau_A = \sqrt{1 + \frac{2\Phi_A}{c^2}}\,\Delta t \simeq \left(1 + \frac{\Phi_A}{c^2}\right)\Delta t$$

Similarly, $\Delta\tau_B = \left(1 + \frac{\Phi_B}{c^2}\right)\Delta t$

$$\Rightarrow \Delta\tau_B = \frac{\left(1 + \frac{\Phi_B}{c^2}\right)}{\left(1 + \frac{\Phi_A}{c^2}\right)}\Delta\tau_A$$

$$\Rightarrow \Delta\tau_B \simeq \left(1 + \frac{\Phi_B}{c^2}\right)\left(1 - \frac{\Phi_A}{c^2}\right)\Delta\tau_A$$

$$= \left(1 + \frac{\Phi_B - \Phi_A}{c^2}\right)\Delta\tau_A + \cdots$$

Same result as before!

III Particle Motion in Spacetime

How do particles move in a curved spacetime?

In classical Mechanics we found a very general principle ~~to~~ that answers this question:

Hamilton's Principle

A particle moves btwn a pt in space at one time and another pt in space at a later time so as to extremize the action in between:

$$S = \int_{t_A}^{t_B} L\, dt \qquad L = T - V$$

$$\delta S = 0 \iff \vec{F} = m\vec{a}$$

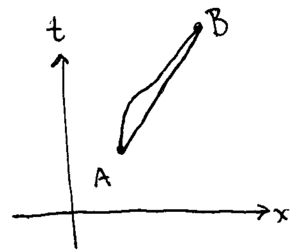
Our geometrical context makes room for a new answer:

Variational Principle for <u>Free</u> Particle Motion

The world line of a free particle btwn two timelike separated pts extremizes the proper time between them.

Let's try this variational principle:

<u>Example</u>: Flat Minkowski spacetime



$$\tau_{AB} = \int_A^B d\tau = \int_A^B \sqrt{-\frac{ds^2}{c^2}}$$

$$= \int_A^B \left(dt^2 - \frac{1}{c^2}\left(dx^2 + dy^2 + dz^2\right)\right)^{1/2}$$

To integrate this over a worldline, we need to choose a parameter along that worldline, call it σ. Then if $\sigma = 0$ corresponds to A and $\sigma = 1$ to B we have,

$$\tau_{AB} = \int_0^1 d\sigma \left(\left(\frac{dt}{d\sigma}\right)^2 - \frac{1}{c^2}\left(\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2 + \left(\frac{dz}{d\sigma}\right)^2\right)\right)^{1/2}$$

This is just like

$$L = \left[\left(\frac{dt}{d\sigma}\right)^2 - \frac{1}{c^2}\left(\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2 + \left(\frac{dz}{d\sigma}\right)^2\right)\right]^{1/2} = \frac{1}{c}\frac{d\tau}{d\sigma}$$

We can use the Euler-Lagrange (E-L) equations to find the worldline that extremizes τ_{AB}:

$$\frac{d}{d\sigma}\left(\frac{\partial L}{\partial(dx^\alpha/d\sigma)}\right) = \frac{\partial L}{\partial x^\alpha} \qquad \left(\text{Recall: } \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}\right)$$

Calculate: What is the E-L eq. for $x^1 = x$?

$$\frac{d}{d\sigma}\left(\frac{\partial L}{\partial(dx/d\sigma)}\right) = \frac{d}{d\sigma}\left(\frac{1}{2}\frac{1}{L}\cdot\left(-\frac{2}{c^2}\frac{dx}{d\sigma}\right)\right) = \frac{\partial L}{\partial x} = 0$$

$$\Rightarrow \frac{d}{d\sigma}\left(\frac{1}{L}\frac{dx}{d\sigma}\right) = 0$$

Same calculation for other coords.s gives

$$\frac{d^2x^\alpha}{d\tau^2} = 0$$

and this is a straight world line.

Now, how about the static weak field metric?

$$\tau_{AB} = \int_A^B d\tau = \int_A^B \left(-\frac{ds^2}{c^2}\right)^{1/2}$$

$$= \int_A^B \left[\left(1+\frac{2\Phi}{c^2}\right)dt^2 - \frac{1}{c^2}\left(1-\frac{2\Phi}{c^2}\right)(dx^2+dy^2+dz^2)\right]^{1/2}$$

But $\frac{1}{L} = \left(\frac{1}{c}\frac{d\tau}{d\sigma}\right)^{-1} = c\,\frac{d\sigma}{d\tau}$

$$\Rightarrow \frac{d}{d\sigma}\left(\frac{d\sigma}{d\tau}\frac{dx}{d\sigma}\right) = \frac{d}{d\sigma}\left(\frac{dx}{d\tau}\right) = 0$$

$$\Rightarrow \frac{d\sigma}{d\tau}\frac{d}{d\tau}\left(\frac{dx}{d\tau}\right) = 0$$

$$\Rightarrow \boxed{\frac{d^2x}{d\tau^2} = 0}$$

The equation of a straight line in the x coordinate.

Let's choose t as our param. this time.

$$\tau_{AB} = \int_A^B dt\left\{\left(1+\frac{2\Phi}{c^2}\right) - \frac{1}{c^2}\left(1-\frac{2\Phi}{c^2}\right)\overbrace{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right]}^{\vec{V}^2}\right\}^{1/2}$$

$$\approx \int_A^B dt\left\{\left(1+\frac{2\Phi}{c^2}\right) - \frac{\vec{V}^2}{c^2}\right\}^{1/2}$$

$$\approx \int_A^B dt\left[1 - \frac{1}{c^2}\left(\frac{1}{2}\vec{V}^2 - \Phi\right)\right]$$