# Lecture 7

Feb $7^{th}$, 2012

## Today's Outline

I Last Lecture

II Particle motion in static weak field metric

III Measurement

IV Coordinates: definitions & bugaboos

## I Last Lecture

- Showed that "slouching clocks run slow" directly from the metric
- Introduced a variational principle for free particle motion:

The worldline of a free particle between two timelike separated points extremizes the proper time between them.

- Free particle in flat spacetime follows a straight world line.

## II Particle motion in Static Weak Field metric

$$\tau_{AB} = \int_A^B d\tau = \int_A^B \left(-\frac{ds^2}{c^2}\right)^{1/2}$$

$$= \int_A^B \left[\left(1+\frac{2\Phi}{c^2}\right)dt^2 - \frac{1}{c^2}\left(1-\frac{2\Phi}{c^2}\right)(dx^2+dy^2+dz^2)\right]^{1/2}$$

Let's choose $t$ as our parameter:

$$\tau_{AB} = \int_A^B dt\left\{\left(1+\frac{2\Phi}{c^2}\right) - \frac{1}{c^2}\left(1-\frac{2\Phi}{c^2}\right)\times \underbrace{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right]}_{\vec{V}^2}\right\}^{1/2}$$

$$\approx \int_A^B dt\left\{\left(1+\frac{2\Phi}{c^2}\right) - \frac{\vec{V}^2}{c^2}\right\}^{1/2}$$

$$\approx \int_A^B dt\left[1 - \frac{1}{c^2}\left(\frac{1}{2}\vec{V}^2 - \Phi\right)\right]$$

So, neglecting the constant $+1$,

$$L = \frac{1}{2}\vec{V}^2 - \Phi(x^i)$$

What are the E-L eq.s?

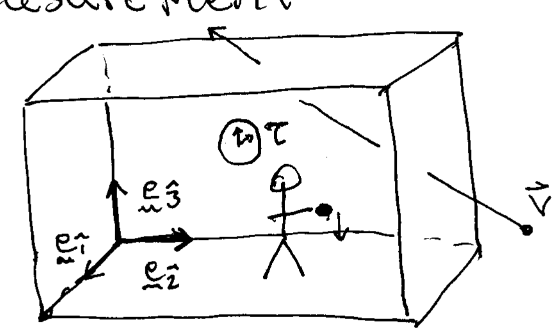
$$\frac{d}{dt}\left(\frac{\partial L}{\partial\left(\frac{dx}{dt}\right)}\right) = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{\partial L}{\partial x} = -\frac{\partial \Phi}{\partial x}$$

$$\Rightarrow \quad \frac{d^2 x}{dt^2} = -\frac{\partial \Phi}{\partial x} \quad !$$

or

$$\frac{d^2 \vec{x}}{dt^2} = -\vec{\nabla}\Phi \quad \text{for all coords.}$$

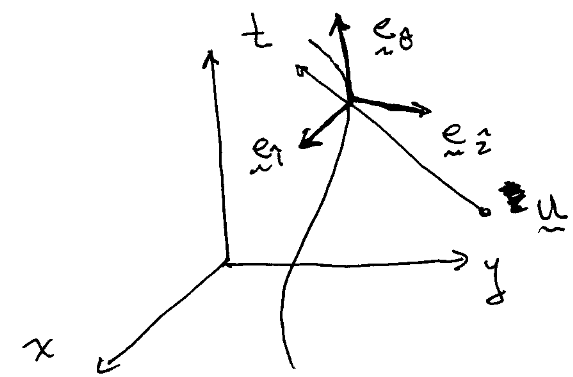
## IV Measurement




Observers make local measurements by setting up axes and clocks: four orthogonal unit vectors $\underset{\sim}{e}_{\hat{0}}$, $\underset{\sim}{e}_{\hat{1}}$, $\underset{\sim}{e}_{\hat{2}}$, $\underset{\sim}{e}_{\hat{3}}$

Interesting remark:

Notice that for both the gravitational time dilation and the recovery of the Newtonian equations of motion it was the curvature in time that was most important; the spatial coefficient $\left(1-\frac{2\Phi}{c^2}\right)$ was irrelevant.




But recall $\underset{\sim}{u}_{obs} \cdot \underset{\sim}{u}_{obs} = -1$, so $\underset{\sim}{u}_{obs}$ is a unit (timelike) vector. So an observer naturally chooses $\underset{\sim}{e}_{\hat{0}} = \underset{\sim}{u}_{obs}$.

To determine what the observer measures we decompose, for example, a particle's four-momentum $\underset{\sim}{p}$ into the observer's coordinates:

$$\underset{\sim}{p} = p^{\hat{\alpha}} \underset{\sim}{e}_{\hat{\alpha}}$$

(think of $\vec{v} = v_x \hat{x} + v_y \hat{y} + v_z \hat{z}$)

The $p^{\hat{\alpha}}$ represent the Energy and Momentum as measured by our moving observer. To calculate these components

we use

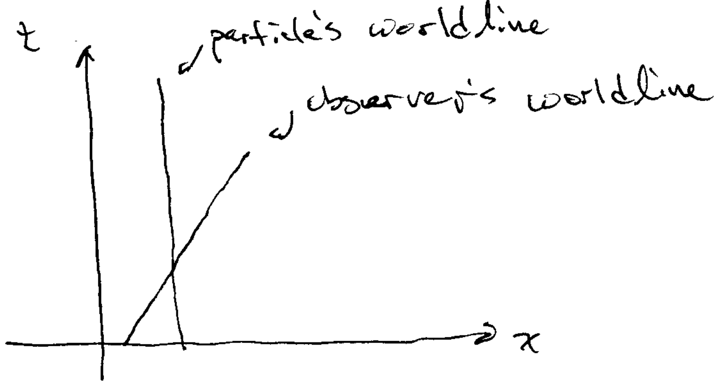
$$\underset{\sim}{e}_{\hat{\beta}} \cdot \underset{\sim}{p} = p^{\hat{\alpha}} \underbrace{\underset{\sim}{e}_{\hat{\beta}} \cdot \underset{\sim}{e}_{\hat{\alpha}}}_{\text{assumption that these are orthonormal}}$$

$$= p^{\hat{\alpha}} \eta_{\hat{\beta}\hat{\alpha}}$$

$$\Rightarrow p^{\hat{0}} = -\underset{\sim}{e}_{\hat{0}} \cdot \underset{\sim}{p}, \quad p^{\hat{1}} = \underset{\sim}{e}_{\hat{1}} \cdot \underset{\sim}{p} \text{ etc.}$$

Lets see it work in practice:

Example: Consider a particle of mass $m$ at rest in an inertial frame. An observe moves along the $x$-axis of this ~~frame~~ frame with velocity $\vec{V}$. What is the energy of the particle according to this observer?




Three Methods: 1) According to the observe the particle moves with ~~s~~velocity $-\vec{V}$ and so

$$E = \gamma m \qquad \gamma = \gamma(\vec{V})$$

2) In the Frame drawn

$$\underset{\sim}{u}_{obs} = (\gamma, \gamma \vec{V}, 0, 0) \qquad \gamma = \gamma(\vec{V})$$
$$= \underset{\sim}{e}_{\hat{0}}$$
$$\underset{\sim}{p} = (m, 0, 0, 0)$$

and so

$$E = -\underset{\sim}{e}_{\hat{0}} \cdot \underset{\sim}{p} = m\gamma \quad \checkmark$$

3) In the observer's frame

$$\underset{\sim}{u}_{obs} = (1,0,0,0) = \underset{\sim}{e}_{\hat{0}}$$

$$\underset{\sim}{p} = (\gamma m, -\gamma m v, 0, 0)$$

and so,

$$E = -\underset{\sim}{e}_{\hat{0}} \cdot \underset{\sim}{p} = \gamma m \quad \checkmark$$

Methods 2) & 3)
~~This~~ illustrates an important point:

Dot products are Lorentz invariants

and so you can evaluate them in whatever coordinates are convenient — you will always get the same answer. Method 1) will become difficult in G.R., methods 2) & 3) become far superior.

## IV Coordinates

Even more than in your past experiences, no one coordinate system is good for all of G.R. Instead we try to extract physically meaningful quantities that are independent of the coordinates used to describe them.

<u>Coords</u>: A unique set of labels for every point in the region they cover.

<u>Examples</u>: Region: Minkowski space

coord.s: $(t, x, y, z)$ $\quad -\infty < t, x, y, z < \infty$
(Range)

or

coord.s $(t, r, \theta, \phi)$
$-\infty < t < \infty$
$0 < r < \infty$
$0 \leq \theta \leq \pi$
$0 \leq \phi \leq 2\pi$

and many, many more.

One reason for our focus on invariants is that the <u>same</u> metric can look very different in different coordinates

Minkowski metric:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

In spherical polar coord.s,

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$dx = dr\, s\theta c\phi + r c\theta c\phi d\theta - r s\theta s\phi d\phi$$
etc.

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$$

If you hadn't seen this transform. a number of times it could be difficult to recognize these as the same metric. More on this shortly.



Metric in general coord.s:

For general coords $x^\alpha$ $\alpha = 0,1,2,3$ we have

$$\boxed{ds^2 = g_{\alpha\beta}(x)\, dx^\alpha dx^\beta}$$

$g_{\alpha\beta}$ is called the metric

<u>Example</u>: flat spacetime in spherical polar coord.s

$$g_{\alpha\beta}(x) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 s^2\theta \end{pmatrix}.$$

$$= \mathrm{Diag}(-1, 1, r^2, r^2 s^2\theta)$$

The metric is symmetric and ~~has~~ so has 10 independent components. Due to coord. transform. 4 of these are arbitrary and so there are 6 true freedoms.

= "A fancied object of terror!"

<u>Bugaboos</u>: Most coordinate systems have problems. For example, in spherical polar coord.s $\theta = 0$ and $\theta = \pi$ for fixed $(t, r)$ don't ~~have label~~ have unique labels (all values of $\phi$ correspond to the same point).

Mathematicians have fixed this up with the notion of a manifold.