Lecture 8

Feb 9th, 2012

Today's Outline

I Last Lecture

II Coordinates: Bugaboos

III Fitting it all together: Local Inertial Frames

IV Vectors in Curved Geometries

I Last Lecture

- Particle motion in weak static field.
- Measurement

  $E = -\mathbf{p}\cdot\mathbf{e}_{\hat{0}} \quad p^{\hat{1}} = \mathbf{p}\cdot\mathbf{e}_{\hat{1}}$
- Coordinates:
  - unique labels for pts in a region
  - general metric: $g_{\alpha\beta}$
  6 independent components

II Coordinates: Bugaboos

Most coordinate systems have issues. For example, in spherical polar coord.s $\theta = 0$ and $\theta = \pi$ for fixed (t, r) don't have unique labels (all values of ϕ correspond to the same point).

Mathematicians have fixed this sort of problem with the notion of a manifold. Another example in the

Plane is,

$$dS^2 = dr^2 + r^2 d\phi^2$$

under the transformation

$r = a^2/r'$, with $a = \text{const.}$

$$dr = -\frac{a^2}{r'^2}dr'$$

$$\Rightarrow \quad dS^2 = \frac{a^4}{r'^4}dr'^2 + \frac{a^4}{r'^2}d\phi^2$$

$$= \frac{a^4}{r'^4}\left(dr'^2 + r'^2 d\phi^2\right)$$

At $r'=0$ the line element blows up. This is our fault for choosing a "bad" coordinate transformation. But it can be (and was) quite confusing at first. As we will see black hole geometries have these <u>coordinate singularities</u>. This caused considerable confusion in the early days of G.R.

<u>The result</u>: Start with a general metric $g_{\alpha\beta}$ in arbitrary coordinates

$$g_{\alpha\beta} = g_{\alpha\beta}(x)$$

Choose a point P of interest then We can always choose new coordinates x' such that

$$g'_{\alpha\beta}(x'_P) = \eta_{\alpha\beta}$$

Q: Can we do even better? A: slightly.

III Local Inertial Frames

The transition to general coordinates is unsettling. How do we use anything from our old tool box (S.R. etc)? The answer to this question is an elegant synthesis of everything we've been thinking about:

We can choose our coord.s s.t.

$$\boxed{g'_{\alpha\beta}(x'_P) = \eta_{\alpha\beta} \quad \text{and} \quad \left.\frac{\partial g'_{\alpha\beta}}{\partial x'^{\gamma}}\right|_{x=x_P} = 0}$$

We call these coordinates a locally inertial frame at P (or Riemann normal coordinates).

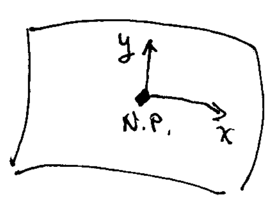
This is precisely what our observers were doing when we discussed a (local) Measurement. This is how special relativity fits into general relativity. This is one more precise formulation of the content of the equivalence principle.

<u>How does it work?</u> As we've noted $g_{\alpha\beta}$ is a symmetric matrix. We choose our coordinate transformation so that this matrix is diagonalized at P other points.

<u>Wonderful Example</u>: We all know that local observers once thought the Earth was flat. What coord.s were they using?

$$dS^2 = a^2(d\theta^2 + \sin^2\theta\, d\phi^2) \quad \text{(sphere)}$$

Slice of sphere near north pole (N.P.)



then by rescaling the new coordinates we can bring the diagonal entries to diag(−1, 1, 1, 1). We'll discuss the additional adjustments necessary for $\partial g'_{\alpha\beta} / \partial x'^{\gamma} \big|_{x_P} = 0$ in a few weeks.

Note that while x' coord.s achieve this at P they do not in general achieve it at

Local observers use

$$x = \underbrace{a\theta}_{\text{arc length}} \underbrace{\cos\phi}_{x\text{-component}}, \quad y = a\theta \sin\phi$$

$$g_{AB}(x,y) = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{flat metric}} + \underbrace{\begin{pmatrix} -\frac{y^2}{3a^2} & \frac{xy}{3a^2} \\ \frac{xy}{3a^2} & -\frac{x^2}{3a^2} \end{pmatrix}}_{\text{1st deriv. vanishes at } (x,y)=(0,0)}$$

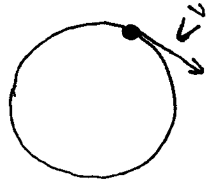
+ higher order

$A, B = (1, 2)$.

Check this!

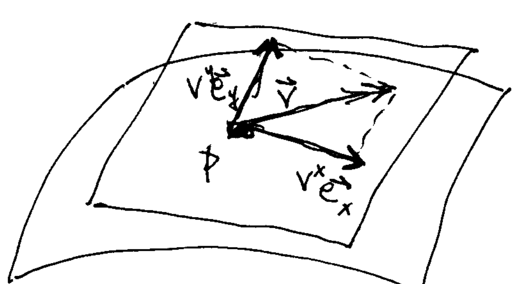
IV Vectors in Curved Geometries

There is a subtlety you are already familiar with but may not have noticed: for curved geometries vectors don't live in position space! Consider a bead on a wire



doesn't live in the circle!

This leads to: 2) where do vectors live? The local construction just described is what mathematicians call a tangent space. Pictorially



vectors live in the tangent space to a point.

We have to address several questions then:

1.) How does a local observer talk about (i.e. measure) vectors?

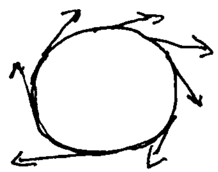
A: The key is to separate directions and magnitudes. Directions are ~~acc~~ accesible locally and then we impose linearity i.e.

$$\alpha(\underset{\sim}{a}+\underset{\sim}{b}) = \alpha\underset{\sim}{a}+\alpha\underset{\sim}{b}.$$

to build up larger magnitude vectors

3) Does this mean that vectors based at different points of our space live in different tangent spaces? Yes! This is important because we can't perform vector operations on vectors that live in different tangent spaces.

Bases: A vector field is an assignment of a vector at each point of your space, e.g.,



We denote this $\mathbf{a}(x)$.

Recall our calculation

$$\mathbf{a}\cdot\mathbf{b} = (a^{\alpha}\mathbf{e}_{\alpha})\cdot(b^{\beta}\mathbf{e}_{\beta}) = a^{\alpha}b^{\beta}(\mathbf{e}_{\alpha}\cdot\mathbf{e}_{\beta})$$

In such a basis,

$$\mathbf{a}\cdot\mathbf{b} = \eta_{\hat{\alpha}\hat{\beta}}\, a^{\hat{\alpha}} b^{\hat{\beta}}$$

As you know, these bases are important because they are the bases of potential observers

$$\mathbf{p} = p^{\hat{\alpha}}\mathbf{e}_{\hat{\alpha}}$$ these components are observable.

Coordinate Basis: We've worked with

We can pick our basis $\mathbf{e}_{\alpha}$ such that the dot product has different characters.

Orthonormal Basis:

A basis for which

$$\boxed{\mathbf{e}_{\hat{\alpha}}(x)\cdot\mathbf{e}_{\hat{\beta}}(x) = \eta_{\hat{\alpha}\hat{\beta}}}$$

hatted indices tell us that we're working with an orthonormal basis.

$$\eta_{\hat{\alpha}\hat{\beta}} = \mathrm{diag}(-1,1,1,1)$$

$$u^{\alpha} = \frac{dx^{\alpha}}{d\tau}$$

Implicitly this was in a particular basis. which one? Well, want $\mathbf{u}\cdot\mathbf{u} = -1$, and for

$$g_{\alpha\beta}u^{\alpha}u^{\beta} = g_{\alpha\beta}\frac{dx^{\alpha}}{d\tau}\frac{dx^{\beta}}{d\tau} = -1.$$

So, these components are in a basis such that,

coordinate basis $\quad \boxed{\mathbf{e}_{\alpha}(x)\cdot\mathbf{e}_{\beta}(x) = g_{\alpha\beta}(x).}$