When can you drop the square root in a Calculus of Variations problem?

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Suppose that you'd like to find the geodesics on a cylinder of radius R. The arc-length functional for this problem is given by,

$$s = \int \sqrt{R^2 d\phi^2 + dz^2}.$$

If we parametrize by some general parameter, call it λ, then this becomes

$$s = \int \sqrt{R^2 \left(\frac{d\phi}{d\lambda}\right)^2 + \left(\frac{dz}{d\lambda}\right)^2} \, d\lambda.$$

We could proceed with calculating the Euler-Lagrange equations from this functional. However, the square root is quite a nuisance, it makes all of the derivatives a pain to calculate. Instead let us choose a very special parameter (we have the freedom to choose any parameter we like) and show that we can simply drop the square root from our calculations all together. Let's choose to parametrize by arc-length, that is by s, then we have,

$$s = \int \sqrt{R^2 \phi'^2 + z'^2} \, ds \qquad \text{with} \qquad ' \equiv \frac{d}{ds}.$$

As short hand let us introduce the following notation,

$$s = \int L \, ds = \int \sqrt{\tilde{L}} \, ds,$$

with $L = \sqrt{R^2 \phi'^2 + z'^2}$ and $\tilde{L} = R^2 \phi'^2 + z'^2$. We'd like to know whether we can drop the square root, that is whether we can simply calculate the Euler-Lagrange equations for $\tilde{L}$ instead of those of L. Well, $L = \sqrt{\tilde{L}}$, so let's calculate Euler's equations for L in terms of $\tilde{L}$,

$$\frac{d}{ds}\left(\frac{\partial L}{\partial q'}\right) - \frac{\partial L}{\partial q} = 0,$$

$$\implies \frac{d}{ds}\left(\frac{\partial \sqrt{\tilde{L}}}{\partial q'}\right) - \frac{\partial \sqrt{\tilde{L}}}{\partial q} = 0.$$

Expanding the second line out using the chain rule,

$$\frac{d}{ds}\left(\frac{1}{2\sqrt{\tilde{L}}}\frac{\partial \tilde{L}}{\partial q'}\right) - \frac{1}{2\sqrt{\tilde{L}}}\frac{\partial \tilde{L}}{\partial q} = 0,$$

and expanding the s derivative using the product rule,

$$\frac{d}{ds}\left(\frac{1}{2\sqrt{\tilde{L}}}\right)\frac{\partial \tilde{L}}{\partial q'} + \frac{1}{2\sqrt{\tilde{L}}}\frac{d}{ds}\left(\frac{\partial \tilde{L}}{\partial q'}\right) - \frac{1}{2\sqrt{\tilde{L}}}\frac{\partial \tilde{L}}{\partial q} = 0.$$

Finally multiplying by $2\sqrt{\tilde{L}}$ leaves us with,

$$2\sqrt{\tilde{L}}\frac{d}{ds}\left(\frac{1}{2\sqrt{\tilde{L}}}\right)\frac{\partial \tilde{L}}{\partial q'} + \frac{d}{ds}\left(\frac{\partial \tilde{L}}{\partial q'}\right) - \frac{\partial \tilde{L}}{\partial q} = 0.$$

Here we have the Euler-Lagrange equation for $\tilde{L}$ except for one extra term, the first term. This is where our special choice of parametrization comes in; notice that the following equalities are true,

$$s = \int ds = \int L ds.$$

But this means that the numerical value of L is one, $L = 1$. But if $L = 1$ then the extra term in the equation above vanishes and we have,

$$\frac{d}{ds}\left(\frac{\partial L}{\partial q'}\right) - \frac{\partial L}{\partial q} = \frac{d}{ds}\left(\frac{\partial \tilde{L}}{\partial q'}\right) - \frac{\partial \tilde{L}}{\partial q} = 0,$$

the Euler-Lagrange equation for L and that for $\tilde{L}$ are the same! In summary, you can drop an overall square root in a calculus of variations problem **only** if you can find a parametrization such that the original Lagrangian is a constant.

This proof is surprisingly subtle, it's worth giving yourself some time to let it sink in. Here are a few exercises you could try to make sure you understand:

1. Argue that you can still drop the square root if you parametrize by any linear transformation of the arc-length, $\lambda = as + b$ with a and b constants.

2. For the geodesics in the plane using polar coordinates: show that you *cannot* drop the square root if you parametrize by r. One approach would be to drop the square root and show that you don't get straight lines.

3. (For the more mathematically inclined.) Generalize the above result to any overall function of the simpler Lagrangian. That is, argue that if $L(q, q') = f(\tilde{L}(q, q'))$ then you can just use $\tilde{L}$ if you have a parametrization such that $L =$const.