Cole on Markov Chains

Markov Chains:

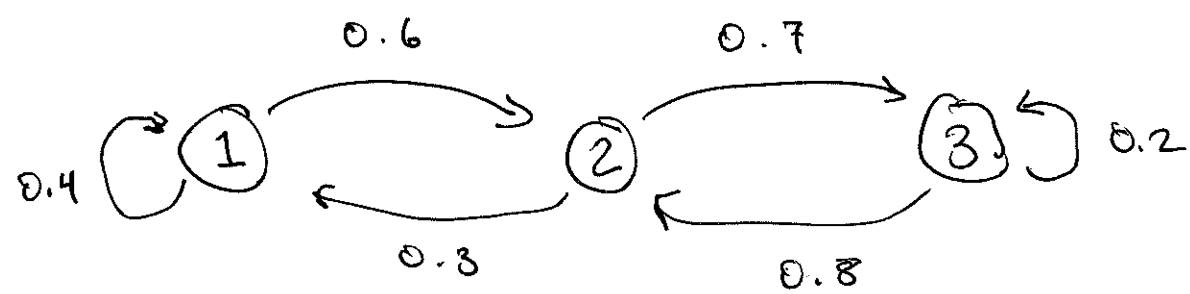
Space of states Ω

Transition probabilities

Markov property = states are memoryless

$$P(X_n \mid X_{n-1} \cdots X_0) = P(X_n \mid X_{n-1})$$

We'll consider Markov chains that have a discrete time variable, and proceed with homogenous time steps = equally sized time steps.

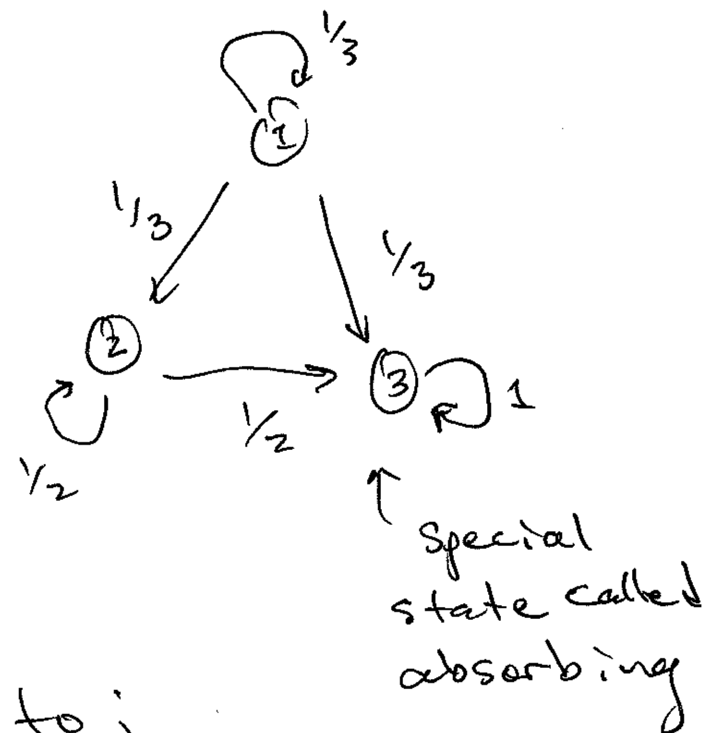


$P(1|1) = 0.4$, $P(1|2) = 0.6$

$$P(3|2)P(2|1) = (0.7)(0.6) = 0.42$$

Transition Matrix

$$P = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$$



The components are denoted

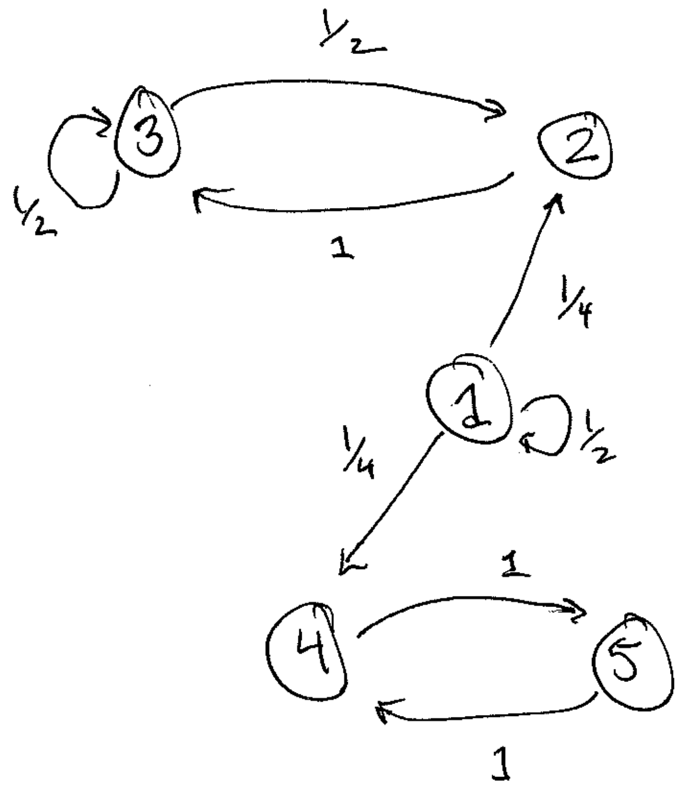
$P_{i,j}$ = prob i jumps to j

The matrix for our first diagram is

$$P = \begin{bmatrix} 0.4 & 0.6 & 0 \\ 0.3 & 0 & 0.7 \\ 0 & 0.8 & 0.2 \end{bmatrix}.$$

Initial probabilities are denoted by ϕ, e.g.

$$\phi(1) = \phi(2) = 1/2$$



if $i \leftrightarrow j$ for all i and j then chain communicates.

The communication classes are

$C_1 = \{2, 3\}$ closed

$C_2 = \{4, 5\}$ closed

$C_3 = \{1\}$ not closed

disjoint membership.

Regular MCs have "regular P"

A regular matrix has all components > 0

e.g. $\begin{bmatrix} 0.2 & 0.8 \\ 1 & 0 \end{bmatrix}$ is regular

but $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is not.

$$\underbrace{\begin{bmatrix} 0.2 & 0.8 \end{bmatrix}}_{\phi} \underbrace{\begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix}}_{P} = \underbrace{\begin{bmatrix} 0.62 & 0.38 \end{bmatrix}}_{\phi_1}$$

Repeating

$$\phi_1 P = \phi_2 = [0.724 \quad 0.276]$$

Eventually

$$\tilde{\phi} = [0.75, \ 0.25]$$

and satisfies $\tilde{\phi} P = \tilde{\phi}$.

This stationarity is a generic property of regular Markov Chains.

The rows of P^k for some large k.