Homework 12

Due Friday, November 18th at 5pm

Reading: Read Boas Ch. 15, sections 1-4.

1. (**a**) Given $f = f(x, y, z)$, and denoting the position vector $\vec{r} = (x, y, z)$, show that
$$df = \vec{\nabla} f \cdot d\vec{r}.$$
Use this property of the gradient to prove the following important results: (**b**) The vector $\vec{\nabla} f$ at any point $\vec{r}$ is perpendicular to the surface of constant f through $\vec{r}$. (Choose a small displacement $d\vec{r}$ that lies in a surface of constant f. What is df for such a displacement?) (**c**) The direction of $\vec{\nabla} f$ at any point $\vec{r}$ is the direction in which f increases fastest as we move away from $\vec{r}$. (Choose a small displacement $d\vec{r} = \epsilon\vec{u}$, where $\vec{u}$ is a unit vector and ϵ is fixed and small. Find the direction of $\vec{u}$ for which the corresponding df is maximum, bearing in mind that $\vec{a} \cdot \vec{b} = ab\cos\theta$.)

2. Boas Ch 4, 8.8. & Boas Ch 4, 8.14.

3. Find the point P on the curve $4x^2 + 3xy = 45$ that is closest to the origin, and then show that the line from the origin through that point P is perpendicular to the tangent line to the curve at P. In light of the method of Lagrange multipliers, why would we expect this result?

4. Optimization with constraints occurs frequently in business settings. For example, if L denotes the number of manhours and K denotes the number of units of capital required to produce q units of a commodity, then q is often related to L and K by a Cobb-Douglas function
$$q = AK^{\alpha}L^{\beta}, \tag{1}$$
where A is a constant, α is the product elasticity of labor, and β is the product elasticity of capital. If one unit of labor costs w dollars and one unit of capital costs r dollars, then the cost to manufacture q units of a commodity is
$$C = wL + rK. \tag{2}$$
Often the number of items to be produced, q is a constant, so that (1) is used as a constraint to (2).

   Cobb and Douglas introduced the idealized production function
$$q = AK^{3/4}L^{1/4}$$
as a model of the interplay of Labor and Capital in the U.S. economy from 1889 to 1929. Find the values of L and K that minimize total cost and determine the **ratio** of labor to capital when total cost is minimized, given that $A = 0.8372$ during that time and that one unit of capital has the same cost as one unit of capital.

5. Boas Ch 4, 13.1.

6. Boas Ch 4, 13.28.