Homework 5

Due Friday, September 30th at 5pm

Reading: Read Boas Ch. 3, section 14, section 13 and reread section 10 and section 9.

1. On Friday we introduced the notion of a basis of a vector space. Choose any basis of $\mathbb{R}^2$ and collect its two vectors into a matrix, call it B, with each vector a column of the matrix. Use a drawing of your two vectors to predict what the determinant of the matrix B is before calculating it. Why does your prediction make sense? Next calculate the determinant of the matrix. Does it agree with your prediction? It's fine if it doesn't, but explain why they agree or disagree.

2. (**a**) Something that we haven't yet emphasized in class is that matrix multiplication is not commutative. That is, in general
$$MN \neq NM.$$
Find an example of two explicit 2×2 matrices M and N where $MN \neq NM$.

   (**b**) In class we came across the useful fact that
$$\det(MN) = \det(NM) = \det M \cdot \det N,$$
with M and N both $n \times n$ matrices. Prove this important fact for one of the orders (MN or NM) by direct computation for the case of 2×2 matrices.

   (**c**) Notice that your result from (**b**) holds independent of the order of multiplication of the matrices. Use your example from (**a**) to confirm that despite the fact that $MN \neq NM$ we still have
$$\det(MN) = \det(NM).$$

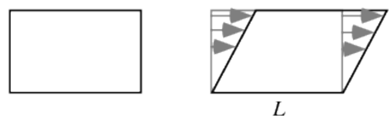
3. (**a**) Using the basis that you chose in problem 1., use each of the transformations
$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad S = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \qquad \text{and} \qquad T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$
to transform both of the vectors of your basis and draw the resulting vectors in each of the cases.

   (**b**) One way of viewing what you've just calculated is as a change of basis with the new matrix of basis vectors, call it B', being given by
$$B' = RB$$
(or $B' = SB$, etc). Use your results from (**a**) to calculate the determinant of B'.

   (**c**) Calculate the determinant of B' again, but this time using the right hand sides of the equations $B' = RB$ (or $B' = SB$, etc).

   (**d**) Use the results of part (**c**) to come up with a conjecture for the interpretation of the determinant of a transformation matrix T. State your conjecture carefully and prove it.

4. The fantastic blog *F&#k Yeah Fluid Dynamics* (fuckyeahfluiddynamics.tumblr.com/) has wonderful images and videos of every sort of fluid phenomenon. When two layers of a fluid are moving at different speeds the fluid in between is subject to a *shearing transformation*. Check out the simulation of this here, a beautiful physical example in this video of Jupiter, and a photograph of clouds doing it too. Before the fluid starts turning over on itself shear transformations are well modeled by a linear transformation.
   (**a**) Find a matrix representation of the shear transformation in this picture:



   (How drastically the transformation shears the rectangle is up to you.)

   (**b**) Is a shear transformation an example of an orthogonal transformation? Why or why not?

5. (**a**) Use your result from problem 3.(**d**) and your matrix representation of a shear transformation to prove that a rectangle and any parallelogram that is a shearing transformation of it have the same area.

   (**b**) In class we argued that the magnitude of the cross product of two vectors $\vec{a}$ and $\vec{b}$ is given by the area of the parallelogram that the two vectors determine, that is, $|\vec{a}\times\vec{b}| = ab\sin\theta$, where θ is the angle between the two vectors. Here you will give a new proof of this fact. Argue that by rotating the two vectors and then shearing one of them as in problem 4. you can transform them so that they form two of the edges of a rectangle. We know that we can calculate the cross product by taking the determinant of the matrix formed by the two vectors. Did the two transformations that you suggest change this determinant? So is the magnitude of the cross product before the transformation equal to that after the transformation?

   What did these two transformations do to the area spanned by the two vectors? (Use what you know of rotations and part (**a**) to answer this question.) What is the area of the final rectangle? So, what is the magnitude of the cross product of the two vectors and what is its geometrical interpretation?

6. (**a**) Are the four vectors

$$\vec{a} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \quad \vec{b} = \begin{pmatrix}1\\1\\0\end{pmatrix}, \quad \vec{c} = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \text{and} \quad \vec{d} = \begin{pmatrix}0\\0\\1\end{pmatrix},$$

   in $\mathbb{R}^3$ linearly independent? If so, how do you know and if not, give the equation for their linear dependency. Can any four vectors in $\mathbb{R}^3$ be linearly independent?

   (**b**) Give a basis for $\mathbb{R}^3$. Prove that the vectors of your basis are linearly independent. [Hint: One way to do this is using proof by contradiction.] What is the dimension of $\mathbb{R}^3$?

7. (**a**) In class we discovered the inverse matrix of a rotation matrix by thinking geometrically. In this problem you will find the general inverse of any 2×2 invertible matrix M. Suppose that the matrix M is given with

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where the values of a, b, c, and d are all known real numbers. We want to find the elements of the matrix M^{-1}, that is the real numbers w, x, y, and z in

$$M^{-1} = \begin{pmatrix} w & x \\ y & z \end{pmatrix},$$

in terms of a, b, c, and d. Use the four equations

$$MM^{-1} = \mathbb{I}$$

to do this. Simplify your result for M^{-1} as much as you can.

(**b**) Use your result from part (**a**) to calculate the inverse matrix R^{-1} of the rotation matrix

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Does your result agree with the geometrical result we found in class?

(**c**) Some matrices M are not invertible. Explain why this can happen using your result from part (**a**). Make a conjecture for the general condition for a matrix not to be invertible? (No need to prove this conjecture right now.) Give an example of a 2×2 matrix with non-zero numerical entries that is not invertible. Act this matrix on a basis of vectors of $\mathbb{R}^2$ and draw the resulting vectors on a diagram. What do you notice?