Math Methods

Day 1

Today

I Why are oscillations so universal?

II Interlude: high-speed video of bouncing ball

III Return to oscillations

IV Review syllabus

Conservation of energy is the key. If the bounce of a ball perfectly conserved energy the ball would return to its initial height. Instead some energy is lost during the bounce as heat. We parametrize this by the coeff. of restitution

$$e = \frac{v_{after}}{v_{before}} = \sqrt{\frac{KE_{after}}{KE_{before}}}$$

I Not because it is a particularly clear example of an oscillation, but because it connects nicely with the mathematics we are about to pick up, let us consider the bouncing of a ball.

Why do many balls bounce? How is it that they return almost to the same height from which they were released?

Using conservation of energy

$$\sqrt{\frac{KE_{after}}{KE_{befor}}} = \sqrt{\frac{PE_{max\ bounce\ height}}{PE_{drop\ height}}}$$

$$= \sqrt{\frac{mgh}{mgH}} = \sqrt{\frac{h}{H}}$$

Then the coeff. e captures the relative height of the bounce

$$e^2 = \frac{h}{H}.$$

II Let's make a high-speed video of this.

III

Another question suggests itself: what is the total distance covered by a bouncy ball when it finally comes to rest? (Assume a constant coeff. $e^2 = \frac{h}{H} = \frac{1}{2}$.)

Suppose we start it at height H and drop it, then the total distance is

rise & fall

∞ terms that follow same pattern

$$D = H + 2\,\tfrac{1}{2}H + 2\,\tfrac{1}{4}H + 2\,\tfrac{1}{8}H + \cdots$$
$$= H + H\left(1 + \tfrac{1}{2} + \tfrac{1}{4} + \cdots\right)$$

We call the infinite sum within the parantheses an <u>infinite series</u>. The $\cdots$ can be vague and so we introduce a notation for such series

$$\left(1 + \tfrac{1}{2} + \tfrac{1}{4} + \cdots\right) = \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n$$

In fact series of the form

$$S = \sum_{n=0}^{\infty} r^n$$

with $r<1$ are so common that they get their own name — they're called geometric series. Remarkably this infinite sum has a finite value

$$S = \frac{1}{1-r}$$

In the homework you'll prove this algebraically and we'll give a geometric proof briefly. For now let's take it as given and calculate with it.

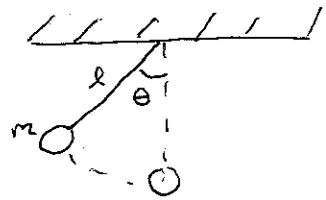
We find

$$D = H + H\left(\frac{1}{1-\frac{1}{2}}\right)$$
$$= H + H\,\frac{1}{(\frac{1}{2})} = 3H\,!$$

A wonderful result. Infinite series give us a way to compute results that would be hard to arrive at otherwise — this is our first motivation for studying them.

Let's continue with our oscillation theme.

Consider a simple pendulum

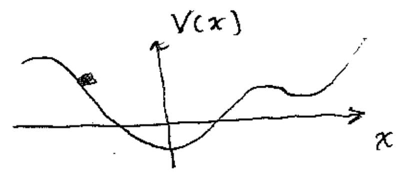


The potential energy of the mass is

$$V = mgh = mgl(1-\cos\theta)$$

Recall that

$$F = -\frac{\partial V}{\partial x}$$



Quite Similarly

$$\underset{\text{torque}}{\tau} = -\frac{\partial V}{\partial \theta}$$

We use this to find

$$\tau = -mgl(\sin\theta)$$

and since $\tau = I\alpha = ml^2\ddot{\theta}$ always

we have

$$\ddot{\theta} = -g/l \sin\theta.$$

This is not a differential equation that belongs to the standard tool box. However, many of you will have seen one way to treat it.

If you use the approximation

$$\sin\theta \approx \theta \quad \text{for small } \theta$$

it becomes the equation of simple harmonic motion

$$\ddot{\theta} \approx -g/l\,\theta \equiv -\omega^2\theta.$$

But what justifies the approximation $\sin x \approx x$ and why is it only valid for small x? Infinite series can be used to answer these questions. In fact they can be used to give a whole new perspective on functions, such as $\sin(x)$, and on their approximations — this is our 2nd motivation for studying them.

Let's try to find a series that equals $\sin x$. This time the series will have to depend on x since $\sin x$ does. So, instead of adding up an infinite list of constants $\{a_n\}$, let's try all powers of x with constant coefficients

$$\sin x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$
$$= \sum_{n=0}^{\infty} a_n x^n$$

To compute a number with this formula we need to calculate all the a's. Let's try. The formula is supposed to hold for all x; let's evaluate both sides at $x=0$, we get

$$\sin(0) = 0 = a_0 + 0 + 0 + \cdots$$

Then $\boxed{a_0 = 0}$ must hold! We'd like to keep using this idea to get a_1, a_2 etc. But how can we isolate a_1 from the x dependence? We can take a derivative of both sides

$$\cos x = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

Then

$$\cos(0) = 1 = a_1 \Rightarrow \boxed{a_1 = 1}$$

Now we can keep repeating

$$-\sin x = 2a_2 + 6a_3 x + \cdots$$

$$\Rightarrow -\sin(0) = 2a_2 \Rightarrow \boxed{a_2 = 0}$$

and

$$-\cos(x) = 6a_3 + \cdots$$
$$\Rightarrow -1 = 6a_3 \Rightarrow a_3 = -\frac{1}{6}$$

Putting this all together

$$\sin x = 0 + x + 0 \cdot x^2 - \frac{1}{6}x^3 + \cdots$$
$$= x - \frac{1}{1 \cdot 2 \cdot 3}x^3 + \cdots$$

in general

$$= x - \frac{1}{3!}x^3 + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

We can see now why the approximation

$$\sin x \simeq x \quad \text{for small } x$$

works. If $x = 0.01 = 10^{-2}$ we have

$$\frac{x^3}{3!} = \frac{1}{6} \times 10^{-6}$$

and the first correction in the series is in the 7th decimal place.

We'll pick up on this discussion next time.

IV Reviewed Syllabus