Math Methods
Day 2

Today

I Last time

II Discuss "What works, What doesn't"

III Power Series & Taylor Series

IV Why are oscillations so universal?

- Reviewed the equation of motion of a pendulum. We found

$$\frac{d^2\theta}{dt^2} \equiv \ddot{\theta} = -\,{}^{g}/_{\ell} \sin\theta$$

$$\approx -\,{}^{g}/_{\ell}\,\theta \quad \text{for small } \theta$$

Motivation 2: Infinite Series provide a transparent justification of this approximation and a whole new perspective on functions in general; we will be able to

I • Introduced the geometric series

$$S = 1 + r + r^2 + r^3 + \dots$$

$$= \frac{1}{1-r}$$

Motivation 1: Infinite Series give us a powerful computational tool. We can find the distance travelled by a ball after an infinite # of bounces.

find a series

$$\sin x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$= \sum_{n=0}^{\infty} a_n x^n$$

a so-called Taylor Series. This is where we'll pick up today.

II Article was ~~-~~ quasi useful.

III Let's try to find a series that equals $\sin x$. This time the series will have to depend on x since $\sin x$ does. So, instead of adding up an infinite list of constants $\{a_n\}$, let's try all powers of x with constant coefficients

$$\sin x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$
$$= \sum_{n=0}^{\infty} a_n x^n$$

To compute a number with this formula we need to calculate all the a's. Let's try. The formula is supposed to hold for all x; let's evaluate both sides at $x=0$, we get

$$\sin(0) = 0 = a_0 + 0 + 0 + \cdots$$

$$\Rightarrow \quad -\sin(0) = 2a_2 \Rightarrow \boxed{a_2 = 0}$$

and

$$-\cos(x) = 6a_3 + \cdots$$
$$\Rightarrow -1 = 6a_3 \Rightarrow a_3 = -\frac{1}{6}$$

Putting this all together

$$\sin x = 0 + x + 0 \cdot x^2 - \frac{1}{6}x^3 + \cdots$$
$$= x - \frac{1}{1 \cdot 2 \cdot 3} x^3 + \cdots$$

in general

$$= x - \frac{1}{3!}x^3 + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Then $\boxed{a_0 = 0}$ must hold! We'd like to keep using this idea to get a_1, a_2 etc. But how can we isolate a_1 from the x dependence? We can take a derivative of both sides

$$\cos x = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

Then

$$\cos(0) = 1 = a_1 \Rightarrow \boxed{a_1 = 1}$$

Now we can keep repeating

$$-\sin x = 2a_2 + 6a_3 x + \cdots$$

We can see now why the approximation

$$\sin x \approx x \quad \text{for small } x$$

works. If $x = 0.01 = 10^{-2}$ we have

$$\frac{x^3}{3!} = \frac{1}{6} \times 10^{-6}$$

and the first correction in the series is in the 7th decimal place.

What made the technique we just used work? Was it special to the $\sin x$ function or could we use it more generally?

It was the fact that each derivative killed another power of x — and completely killed the constants. But, then this should work for any power series. We can even shift the powers by a,

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots + a_n(x-a)^n + \cdots$$

Then $\boxed{f(a) = a_0}$

$$f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots + na_n(x-a)^{n-1} + \cdots$$

and so

$$f'(a) = a_1$$

$$f''(a) = 2a_2$$

$$f'''(a) = 6a_3$$

$$f^{(n)}(a) = n!\,a_n.$$

This means we can write $f(x)$ as

$$f(x) = f(a) + (x-a)f'(a) + \frac{1}{2!}(x-a)^2 f''(a) + \cdots$$

This is called a Taylor Series. This is an incredibly useful formula that will come up in every physics course you take. Memorize it!

Let's use this to answer our question why are oscillations so universal?

IV. Simple harmonic motion (SHM):

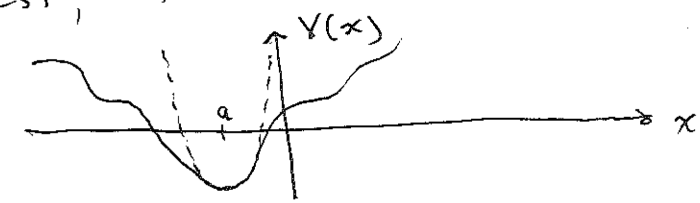
$$x(t) = A\cos(\omega t + \phi)$$

"Force": $\ddot{x} = -\lambda x$, for some $\lambda > 0 \Rightarrow \omega = \sqrt{\lambda}$

"Energy": $a\dot{x}^2 + bx^2 = c$ $(a, b > 0;\ c \text{ constant}) \Rightarrow \omega = \sqrt{\frac{b}{a}}$

(Think of $\frac{1}{2}mv^2 + \frac{1}{2}kx^2$ and $\omega = \sqrt{k/m}$, so $\omega = \sqrt{b/a}$.)

Practically anything that oscillates at all is SHM, at least, for small oscillations.



<u>Proof</u>'': Taylor expand $V(x)$ about the minimum

$$V(x) = V(a) + \cancel{V'(a)}^{0}(x-a) + \frac{1}{2}V''(a)(x-a)^2 + \underbrace{\frac{1}{3!}V'''(a)(x-a)^3 + \cdots}$$

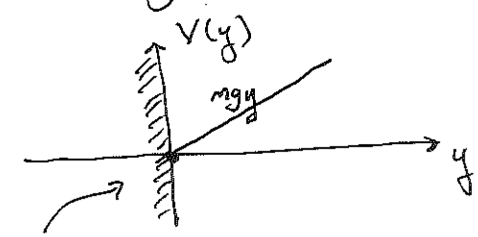
We can ignore higher order terms for small $(x-a)$!

$$\approx V(a) + \frac{1}{2}\underbrace{V''(a)}_{\text{plays role of effective } k}(x-a)^2$$

Basically the best parabola fit.

$$\underbrace{E - V(a)}_{c} = \underbrace{\frac{1}{2}m\dot{x}^2}_{a} + \underbrace{\frac{1}{2}V''(a)(x-a)^2}_{b}$$

<u>Counter example</u>: Our bouncing ball!



Below the floor the potential energy is infinite.

Can't do Taylor expansion because it's not differentiable.

So, "practically" means first that you are able to Taylor expand and secondly that the second derivative at a is not zero.

Motivation 3: Power Series provide a flexible tool for studying the structure of functions.