Math Methods

Day 3

Sep 2nd, 2016

Today

I Last time

II When can we not Taylor expand a function?

III Geometric proof of the geometric series sum

IV Convergence tests

At $x = a$ we have

$$f(x) = f(a) + f'(a)(x-a) + \tfrac{1}{2} f''(a)(x-a)^2 + \dots + \tfrac{1}{n!} f^{(n)}(a)(x-a)^n + \dots$$

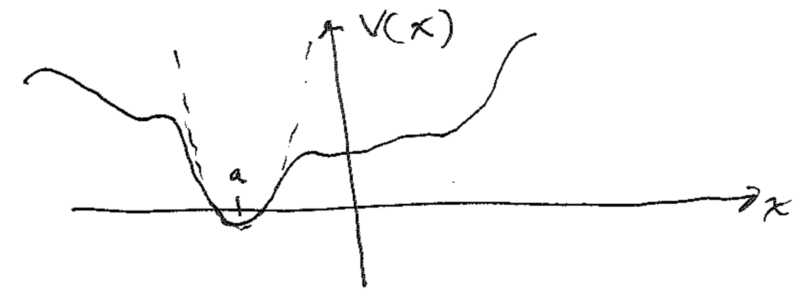
$$= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

- From the Taylor series expansion of a general potential around one of its minima, you can immediately see why simple harmonic motion is so universal

I. • A power series is any series of the form

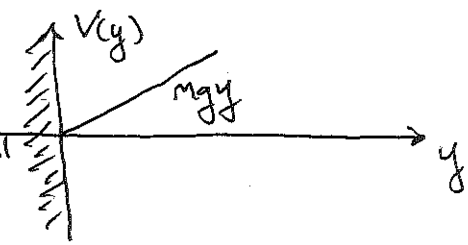
$$S(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$
$$= \sum_{n=0}^{\infty} a_n x^n$$

- A Taylor series is the power series expansion of a function about an arbitrary point of its domain:



II Our bouncing ball provides a counter example to the "proof" from last time:

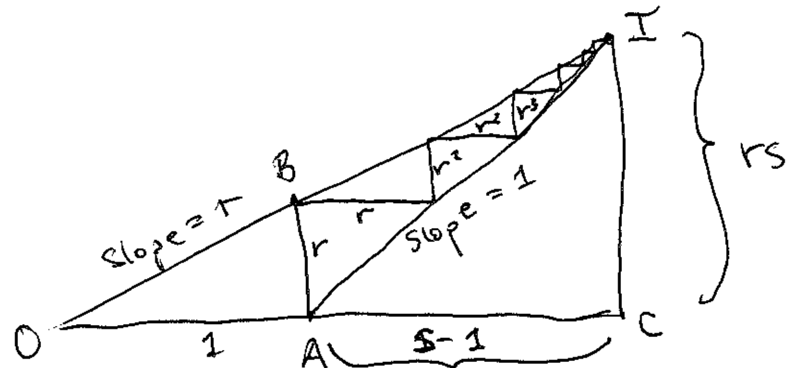
Below the floor the potential is essentially infinite.



We can't do a Taylor expansion about the point $y=0$ because the potential isn't differentiable there. So the conditions for our "proof" to hold are

1. that you are able to Taylor expand (the derivatives of the function exist at the point a.) and
2. The 2^{nd} derivative at a is not zero.

Extend the hypotenuse OB. From A draw a line of slope equal to 1 and intersect these two lines

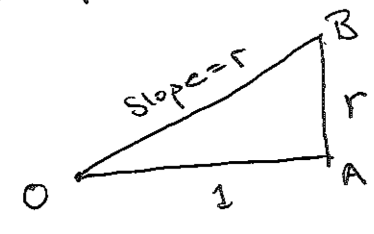


Construct the series of right triangles depicted above

III I promised a geometric proof of

$$S = 1+r+r^2+r^3+\cdots+r^n+\cdots = \frac{1}{1-r}.$$

Construct a triangle with base of length 1 and hypotenuse of slope r:



Let the base of the large triangle have a length S, Then by adding the lengths of all the bases of the small triangles

$$S = 1 + r + r^2 + r^3 + r^4 + \cdots$$

Now since OI has slope r we have $CI = rS$ and because AI has slope 1 we have

$$rS = S-1$$

or

$$\Rightarrow \boxed{S = \frac{1}{1-r}.}$$

IV Definitions:

- If
$$\lim_{n \to \infty} S_n = S$$
is a finite number, then the series S is convergent. Otherwise it is divergent.
- S is the sum of the series.
- The difference
$$R_n = S - S_n$$
is the remainder.

For example if $S = \sum_{n=1}^{\infty} \frac{n!}{n!+1}$ then

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n!}{n!+1} = \lim_{n \to \infty} \frac{n!}{n!} = 1,$$

so the series is $\boxed{\text{divergent.}}$

Four tests for positive series:

If all the terms in a series are positive one can apply the following tests. In fact, they are also usefull for series with negative terms — the strategy is to apply the test to the absolute values of the terms.

In general taking the limit
$$\lim_{n \to \infty} S_n$$
can be difficult, so we'd like simpler tests for convergence.

Preliminary test: If
$$\lim_{n \to \infty} a_n \neq 0$$
the series $\boxed{\text{diverges}}$. But,
$$\lim_{n \to \infty} a_n = 0$$
is inconclusive (i.e. useless).

If the series of absolute values converges we say that the series is absolutely convergent.

You will prove on the homework that if a series converges absolutely then it converges.

Comparison Test: Say you are considering the series

$$a_1 + a_2 + a_3 + a_4 + \cdots$$

To run a comparison test you pick a second series that you know converges

$$m_1 + m_2 + m_3 + m_4 + \cdots$$

and the initial series is absolutely convergent if $|a_n| \le m_n$, $\forall n$ after some n.

Similarly if

$$d_1 + d_2 + d_3 + d_4 + \cdots \qquad d_n > 0 \ \forall n$$

diverges and $|a_n| \ge d_n$ for all n from some point on, then the a_n series diverges.

<u>Integral Test</u>: If $0 < a_{n+1} \le a_n$ for $n > N$, then $\Sigma^\infty a_n$ converges if

$$\int^\infty a_n \, dn \quad \leftarrow \text{means only evaluate at upper limit}$$

is finite and diverges if the integral is infinite.

<u>Example</u>: $\sum_{n=1}^{\infty} \frac{1}{n}$ the harmonic series.

Well, $\int^\infty \frac{1}{n} dn = \ln n \Big|^\infty = \infty$

So the series diverges.

<u>Example</u>:

$$\sum_{n=2}^{\infty} \frac{1}{\ln n}$$

What's a good comparison series? They've argued a few times in our text that $n > \ln n$ for large enough n. This implies that

$$\frac{1}{n} < \frac{1}{\ln n}$$

for large enough n, and since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent our series $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ must also be so.

We clearly are going to need a way to remember all these tests

Po' CIRCuS gu ACC
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Po' = Preliminary

CIRCuS = Comparison, Integral, Ratio, Comparison (Special)

Alternating

Conditional Convergence