# Math Methods

## Day 4

Sep 5th, 2016 

**Today**

I Last time & Regrade

II Finish convergence Tests

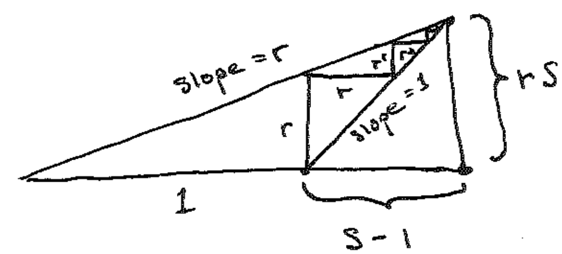
III Alternating Series & Conditional Convergence

IV A Few Useful Facts

I. • Proved the geometric series formula

$$S = \frac{1}{1-r} \quad |r| < 1$$

using




• $\lim_{n\to\infty} S_n = S$, $S$ finite

Convergent. Else divergent.

$S$ is the sum of the series.

• Preliminary test

$\lim_{n\to\infty} a_n \neq 0$ Divergent

Else inconclusive.

• Comparison test

• Integral test: $\int^{\infty} a_n \, dn = \begin{cases} \text{finite} \to \text{convergent} \\ \text{infinite} \to \text{divergent.} \end{cases}$

II The Ratio Test: For the geometric series we have

$$\frac{a_{n+1}}{a_n} = r,$$

and converges as long as $|r| < 1$.

So, let's consider in general

$$S = \sum_{n=0}^{\infty} a_n$$

and

$$\rho_n = \left|\frac{a_{n+1}}{a_n}\right|,$$

with $\rho = \lim_{n\to\infty} \rho_n$.

On the homework we will prove

If $\begin{cases} \rho < 1, & \text{the series converges;} \\ \rho = 1, & \text{Use a different test;} \\ \rho < 1, & \text{The series diverges;} \end{cases}$

Example: $\sum_{n=0}^{\infty} \frac{e^n}{\sqrt{n!}}$, then

$$\rho_n = \frac{e^{n+1}}{\sqrt{(n+1)!}} \cdot \frac{\sqrt{n!}}{e^n} = \frac{e \cdot \sqrt{n!}}{\sqrt{(n+1)\, n!}} = \frac{e}{\sqrt{n+1}}$$

and

$$\rho = \lim_{n\to\infty} \frac{e}{\sqrt{n+1}} = 0 \Rightarrow \boxed{\text{Series converges}}$$

## A Special Comparison Test:

(a) If $\sum_{n=1}^{\infty} b_n$ is a convergent series of positive terms and $a_n \geq 0$ and $a_n/b_n$ tends to a (finite) limit, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\sum_{n=1}^{\infty} d_n$ is a divergent series of positive terms and $a_n \geq 0$ and $a_n/d_n$ tends to a limit greater than 0 (or tends to $+\infty$), then $\sum_{n=1}^{\infty} a_n$ diverges.

Ex: $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+5n-1}}{n^2 - \sin n^3}$; First let's identify the fastest growing piece. In the numerator it is clearly $\sqrt{n^3}$. In the denominator note that $\sin n^3$ is stuck between $-1$ and $+1$, so $n^2$ dominates.

So, we choose as our comparison series $\sum_{n=1}^{\infty} \frac{\sqrt{n^3}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

Let's test this series' convergence. The ratio test gives

$$\rho_n = \left| \frac{1}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{1} \right| = \left| \frac{1}{\sqrt{1 + 1/n}} \right|$$

and

$$\rho = \lim_{n\to\infty} \left| \frac{1}{\sqrt{1+1/n}} \right| = \left| \frac{1}{\sqrt{1}} \right| = 1 \quad \text{inconclusive.}$$

The integral test might work too,

$$\int^{\infty} \frac{1}{\sqrt{n}}\, dn = 2\sqrt{n} \Big|^{\infty} = \infty$$

We see that the series <u>diverges</u>.

Now, we can return to our initial series and consider

$$\frac{\sqrt{n^3+5n-1}}{n^2 - \sin n^3} \cdot \frac{n^2}{\sqrt{n^3}} = \frac{\sqrt{1 + 5/n^2 - 1/n^3}}{1 - \frac{\sin n^3}{n^2}}$$

Finally,

$$\lim_{n\to\infty} \frac{\sqrt{1 + 5/n^2 - 1/n^3}}{1 - \frac{\sin n^3}{n^2}} = \frac{\sqrt{1}}{1} = 1 \Rightarrow \underline{\text{series diverges}}.$$

III We have been focusing on series* of positive terms. But, in studying Taylor series we have noticed that some series are alternating, that is, each term switches sign; a good example is

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots + \frac{(-1)^{n+1}}{n} + \cdots$$

Is this series convergent? Let's start with absolute convergence, is

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$$

A series like this, which converges, but not absolutely, is called <u>conditionally convergent</u>.

The sum of these series depends on the order in which you add the terms. Physical example


Positive charges at $x = 0, \sqrt{2}, \sqrt{4}, \sqrt{6}, \sqrt{8}, \ldots$
negative charges at $x = 1, \sqrt{3}, \sqrt{5}, \sqrt{7}, \ldots$

convergent? No! It's the harmonic series. But, this doesn't yet answer the question about the alternating series.

<u>Alternating Series Test</u>: An alternating series converges if the absolute value of the terms decreases steadily to zero, that is, if $|a_{n+1}| \le |a_n|$ and $\lim_{n\to\infty} a_n = 0$.

For the example, $\frac{1}{n+1} < \frac{1}{n}$ and $\lim_{n\to\infty} 1/n = 0$, so <u>this alternating series converges.</u>

The total force on the charge at the origin is

$$F = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

If the charges are placed one after the other, say by one crew, this is the force. However, if the charges are not placed at the same rate the net force can have a different value.