Math Methods

Day 5

Today

I Last time, OH reminder & location announcement, regrade

II Useful Facts

III Applications to power series

I

- Ratio test

$$\rho_n = \left| \frac{a_{n+1}}{a_n} \right|, \quad \rho = \lim_{n\to\infty} \rho_n$$

If $\begin{cases} \rho < 1 & \text{converges} \\ \rho = 1 & \text{another test} \\ \rho > 1 & \text{diverges.} \end{cases}$

- Special comparison

A combination of the comparison test and the ratio test.

- Alternating series test

An alternating series converges if

$$|a_{n+1}| \le |a_n|$$

and

$$\lim_{n\to\infty} a_n = 0.$$

- Conditional convergence is when a series converges, but not absolutely. Have to be careful with the sums of these series.

II Useful Facts:

- Multiplying a series by a constant does not affect its convergence or divergence.
- We can add convergent series and obtain a convergent series with sum

$$S = S_1 + S_2.$$

Similarly with subtraction of series.

- For absolutely convergent series we can reorder the terms, but this is not so for conditionally convergent series.

III You derived the power series for $\sin x$,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^{n+1} x^{2n-1}}{(2n-1)!} + \cdots$$

For what values of x does this series converge? The ratio test is perfect here:

$$\rho_n = \left| \frac{(-1)^{n+2} x^{2n+1}}{(2n+1)!} \cdot \frac{(2n-1)!}{(-1)^{n+1} x^{2n-1}} \right|$$

$$= \left| \frac{x^2 \ (2n-1)!}{(2n+1)(2n)(2n-1)!} \right| = \left| \frac{x^2}{(2n+1)(2n)} \right|$$

and

$$\rho = \lim_{n\to\infty} \rho_n = 0,$$

for any finite x. So this series converges for <u>all x</u>. This is not true of all power series — we call the range of x values for which a power series converges its <u>interval of convergence</u>.

Let's do a few more examples:

$$\sum_{n=1}^{\infty} (-1)^n n^3 x^n$$

Again we'll try the ratio test,

$$\rho_n = \left| \frac{(-1)^{n+1} (n+1)^3 x^{n+1}}{(-1)^n (n)^3 x^n} \right|$$

$$= \left| (1+\tfrac{1}{n})^3 x \right|$$

So,

$$\rho = \lim_{n\to\infty} \rho_n = |x|$$

and the series certainly converges for $|x|<1$

We need to check the points $|x|=1$ separately, i.e. $x=+1$ and $x=-1$.

So, for $x=-1$,

$$\sum_{n=1}^{\infty} (-1)^n n^3 (-1)^n = \sum_{n=1}^{\infty} n^3,$$

which is not convergent, e.g. by comparison. And $x=+1$ gives

$$\sum_{n=1}^{\infty} (-1)^n n^3,$$

which is alternating. But the terms don't get smaller, so it does not converge. We can conclude that the interval of convergence is

$$|x|<1 \quad \text{or} \quad \boxed{-1<x<1.}$$

Another,

$$\sum_{n=1}^{\infty} \frac{n(-x)^n}{n^2+1}$$

has

$$\rho_n = \left| \frac{(n+1)(-x)^{n+1}}{(n+1)^2+1} \cdot \frac{(n^2+1)}{n(-x)^n} \right|$$

$$= \left| \frac{(1+\frac{1}{n})x}{\frac{(n+1)^2}{(n^2+1)} + \frac{1}{(n^2+1)}} \right|$$

So,

$$\rho = |x|$$

and the series converges for $|x|<1$.

The endpoints? At $x=1$

$$\sum_{n=1}^{\infty} \frac{n(-x)^n}{n^2+1} = \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$$

is alternating. We have

$$\left| \frac{(-1)^{n+1}(n+1)}{(n+1)^2+1} \right| < \left| \frac{(-1)^n n}{n^2+1} \right|$$

and $\lim_{n\to\infty} \frac{(-1)^n n}{n^2+1} = 0$, so it converges! At $x=-1$

$$\sum_{n=1}^{\infty} \frac{n(-x)^n}{n^2+1} = \sum_{n=1}^{\infty} \frac{n}{n^2+1}.$$

We know the ratio test will give $\rho=1$, so let's try the integral test, indeed $a_{n+1} < a_n$,

$$\int^{\infty} \frac{n}{n^2+1}\,dn = \frac{1}{2}\ln(1+n^2)\Big|^{\infty}$$

$$= \infty$$

and the series diverges there.

So, the interval of convergence is

$$|x|<1 \to \boxed{-1<x\le 1.}$$

Within the interval of convergence a power series can be used to define a function

$$S(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Much like for infinite series there are several useful facts about power series:

• You can take derivatives or integrals of power series term by term and will converge on the same interval, except possibly at end points.

<u>Ex</u>: $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$

$$\frac{d}{dx} \sin x = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots \checkmark$$

or

arbitrary constant fixed by bndry condition. $= -1$.

$$\int \sin x = -\cos x = C + \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots + \frac{(-1)^n x^{2n+2}}{(2n+2)!} + \cdots \checkmark$$

• You can add, subtract & multiply power series. The result converges in the common interval of convergence at least. You can divide power series when it makes sense (i.e. no zero denominator). Test for convergence interval.

• You can substitute one series into another when it makes sense (i.e. the values of the subbed series are in the interval of convergence of the other series.)

• The power series of a function is unique.