0. Quiz
I Last time

Math Methods

Day 6

Sept. 9th, 20167 1/3

II Shortcuts for finding power series expansions

- You can add, subtract & multiply power series. Divide too, when it makes sense.
- You can substitute one series into another when it makes sense.
- The power series of a function is unique.

II Choose a polynomial. Choose a function whose power series we know. What is the power series

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- The ratio test is can extremely useful way to find the interval of convergence of a power series

Useful facts about power series:

- Can take derivatives and integrals term-by-term.

of the product (first few terms is fine).

$$(1+2x+3x^3)\,e^x$$

$$= (1+2x+3x^3)\left(1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\cdots\right)$$

$$= 1+x+2x+\frac{x^2}{2}+2x^2+\frac{x^3}{3!}+x^3+3x^3+\cdots$$

$$= 1+3x+\frac{5}{2}x^2+\frac{25}{6}x^3+\cdots$$

Works for the product of two series also, e.g.,

$$e^x \ln(1+x) = \left(1+x+\frac{x^2}{2}+\cdots\right)\left(x-\frac{x^2}{2}+\frac{x^3}{3}+\cdots\right)$$

$$= x - \frac{x^2}{2} + x^2 + \frac{x^3}{3} - \frac{x^3}{2} + \frac{x^3}{2} + \cdots$$

$$= x + \frac{1}{2}x^2 + \frac{x^3}{3} + \cdots$$

We can also divide series

$$\frac{1}{x}\sin x = \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

$$= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}.$$

We can even do long division of series. Recall how we do this for numbers

$$\frac{1}{300} = \begin{array}{r} 0.00333\cdots \\ 300\,\overline{)\,1.000\cdots} \\ -\underline{\;\;900} \\ 1000 \\ 900 \end{array}$$

We can use a similar procedure for series. Take

$$\sec x = \frac{1}{\cos x}$$

as an example. Expanding $\cos x$ we have

$$\begin{array}{r} 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \cdots \\ \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)\overline{)\,1 \qquad\qquad\qquad} \\ -\underline{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)} \\ \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots \\ -\underline{\left(\frac{1}{2}x^2 - \frac{x^4}{4} + \cdots\right)} \\ \frac{5}{24}x^4 + \cdots \end{array}$$

Cool, no?

On the quiz you just derived the 4th of our essential power series. Let's do the last one too. Find the Taylor series of $(1+x)^p$, where p is any real number, positive or negative.

By the chain rule

$$\frac{d}{dx}(1+x)^p = p(1+x)^{p-1}\cdot 1\,\Big|_{x=0} = p$$

$$d^2/dx^2\,(1+x)^p = p(p-1)(1+x)^{p-2}\Big|_{x=0} = p(p-1)$$ and so on,

Then

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$$

If p was an integer then the numerators would look the beginnings of factorials also. In fact, like

$$\frac{p!}{n!\,(p-n)!} = \frac{p(p-1)\cdots(p-(n-1))}{n!}$$

But this is a very well known and important function $\binom{p}{n} = \frac{p!}{n!\,(p-n)!}$

Examples:

$$\frac{1}{1+x} = (1+x)^{-1} \overset{p=-1}{=} 1 - x + x^2 - x^3 + \cdots$$

and

$$\sqrt{1+x} = (1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots$$

This pops up in myriad physical applications. As an example, in special relativity a new definition for energy arises

called the "p choose n" function, or the binomial coefficient. It turns out that this function is also well defined when p is any real number by

$$\binom{p}{n} = \frac{p(p-1)\cdots(p-n+1)}{n!}$$

This series is surprisingly useful in practice.

$$E = \gamma m c^2 \quad \text{where } \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

To see how this relates to our standard notion of energy in mechanics we can expand this as a series in $x = -v^2/c^2$.

Then

$$E = mc^2\left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

$$= mc^2\left(1 - \frac{1}{2}x + \cdots\right)$$

$$= mc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \cdots\right) = mc^2 + \frac{1}{2}mv^2 + \cdots$$