Math Methods Day 7 Sept. 12^{th}, 2016

I Last time

II Foundations of geometry & Numbers

III Complex Numbers

I • We used a variety of techniques to find power series expansions: multiplication of series, long division, etc.

• Found a generalized binomial expansion

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$$
$$= \sum_{n=0}^{\infty} \binom{p}{n} x^n$$

Exs.: $\frac{1}{1+x} = (1+x)^{-1} \overset{p=-1}{=} 1 - x + x^2 - x^3 + \cdots$

and $\sqrt{1+x} = (1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots$

We started to apply this to a physical example – the definition of energy in special relativity.

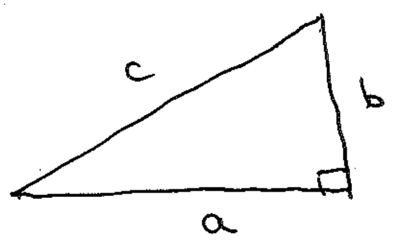
According to Einstein

$$E = \gamma mc^2 \quad \text{with} \quad \gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}$$

In most mechanical systems $v \ll c$ and we can take $x = -v^2/c^2$ as a small parameter, so that

$$E = mc^2\left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$
$$= mc^3\left(1 - \frac{1}{2}x + \cdots\right)$$
$$= mc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \cdots\right)$$
$$= \underbrace{mc^2}_{\text{Rest energy}} + \underbrace{\frac{1}{2}mv^2}_{\text{standard kinetic energy}} + \underbrace{\cdots}_{\text{corrections}}$$

II The Greeks use geometry for practical measurements in land surveying and to explore what is possible. Of great importance to the Pythagoreans was the famous pythagorean formula



$$a^2+b^2=c^2$$

Simplifying the right-hand side gives

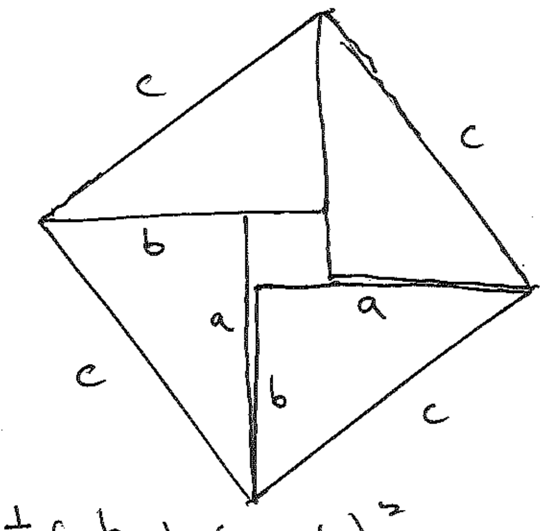
$$c^2 = 2a\cdot b + a^2 - 2ab + b^2$$
$$= a^2+b^2 \ !$$

However, the Greek's focused on integer numbers, like $1, 2, 3, 4, \ldots$, and on rational numbers, like

$$7/8, \ 2/3, \ \frac{1}{2} \ \text{ or } \ \frac{n}{m}$$

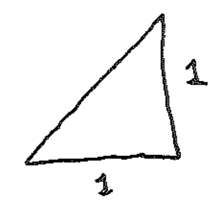
with n and m integers. In symbols we write $n, m \in \mathbb{N}$ for this.

There is a lovely geometrical proof of this



$$c^2 = 4\cdot\tfrac{1}{2}a\cdot b + (a-b)^2$$

This focus led to great concern about the simple triangle



and in particular the length c of its hypotenuse. By the pythagorean formula

$$c^2 = 1+1 \Rightarrow c = \sqrt{2}$$

Is this number an integer or rational

number? The answer turns out to be No! The Greeks realized this and it greatly worried them. We represent $\sqrt{2}$ by its decimal expansion

$$\sqrt{2} = 1.414213562\ldots,$$

but they didn't have this tool. They were very shrewd about ratios though and eventually found an intriguing infinite fraction expansion for $\sqrt{2}$, namely,

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}$$

If we truncate to the first few terms we find

$$\sqrt{2} \approx 1 + \cfrac{1}{2 + \cfrac{1}{2 + \frac{1}{2}}} = 1 + \cfrac{1}{2 + \frac{2}{5}}$$

$$= 1 + \frac{5}{12} = 1.416667 \quad \text{close.}$$

This is a neat way to calculate squareroots by hand and turns out to be applicable to any quadratic number, i.e. numbers of the form

$$a + \sqrt{b}$$

where a, b are rationals and b is not a perfect square. Numbers of this form arise as solutions of the quadratic equation with integer coefficients.

Eventually the Greeks had to accept the extension of their number system by the irrational numbers, those like $\sqrt{2}$.

Today we are going to face a similar dilemma. We want to ask: what happens to our number system when we add the solution of the equation $x^2 = -1$ to it?

III To begin we give the solution a name and define

$$i \equiv \sqrt{-1},$$

which is called the imaginary unit. There is nothing *imaginary* about it, just as there is nothing irrational about $\sqrt{2}$. But it does provide an *extension* of the real numbers, just as $\sqrt{2}$ did for the integers and rationals.

Note again the unfortunate name, the imaginary part of z, $\operatorname{Im} z = y$, is a real number.

For two complex numbers $z = x + iy$ and $w = u + iv$ the sum is

$$z + w = x + u + i(y + v)$$

and similarly for $z - w$. *Ex*: So, if $z = 4i$, $w = 2 - i\sqrt{2}$ then

$$z + w = 2 + i(4 - \sqrt{2}).$$

A complex number is any number of the form

$$z = x + iy$$

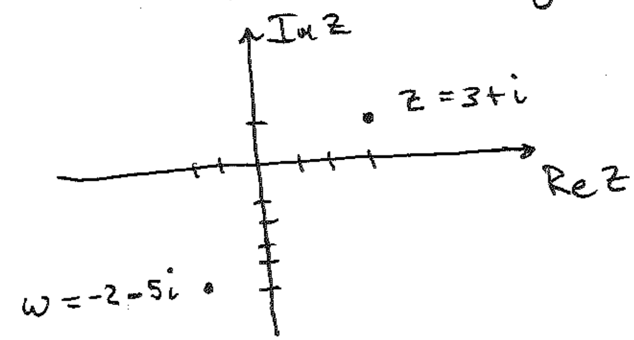
where x and y are real numbers. We call x the "real part of z" and write

$$\operatorname{Re} z = x$$

and y the "imaginary part of z" with

$$\operatorname{Im} z = y$$

Graphically, we display



The somewhat odd notation $z = x + iy$ is justified by the definition of the product for complex numbers via the standard FOIL process (= First + Outer + Inner + Last):

So if $z = x + iy$ and $w = u + iv$ then

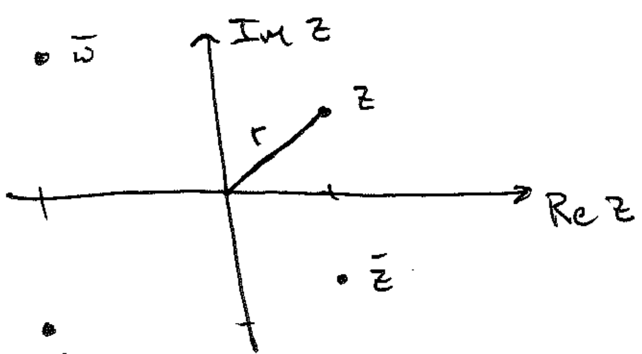
$$z \cdot w = (x+iy)(u+iv)$$
$$= xu + iyu + ixv - yv$$
$$= xu - yv + i(yu + xv).$$

Ex.: $z = 3i$, $w = 2 - i/2$

$$z \cdot w = 6i - \frac{3}{2} i^2 = 6i + \frac{3}{2}$$
$$= 3/2 + i6.$$

or $z = 2+2i$, $w = 1-i$

$$z \cdot w = (2+2i)(1-i) = 4 + 2i - 2i = 4.$$



Now, we have

$$z\bar{z} = (x+iy)(x-iy)$$
$$= x^2 + y^2$$

which is always a positive real number. In fact on the diagram

To define division we need one more operation, which is called complex conjugation. The complex conjugate of $z = x+iy$ is

$$\bar{z} = x - iy,$$

that is you map

$$i \mapsto -i$$

throughout the expression. This gives a reflection about the real axis:

it specifies the radial distance to the point z, so we call it

$$z\bar{z} = x^2 + y^2 = r^2.$$

Because it is always real and positive we use this to define the absolute value of a complex number

$$|z| \equiv \sqrt{z\bar{z}} = \sqrt{r^2} = r.$$

This provides a neat way to define the division of complex numbers:

if $z = x + iy$ and $w = u + iv$

$$\frac{z}{w} = \frac{z}{w}\frac{\bar{w}}{\bar{w}} = \frac{z\bar{w}}{w\bar{w}} = \frac{z\bar{w}}{|w|^2}$$

← a real number

and we've reduced complex division to complex multiplication.

Try Ex:

$$\frac{1}{i} = \frac{1}{i}\frac{\bar{i}}{\bar{i}} = \frac{1\cdot(-i)}{i\cdot(-i)} = \frac{-i}{1} = -i$$

Is this right?

$i \cdot \frac{1}{i}$ should equal 1

and

$$i(-i) = 1$$

Very cool.