Math Methods

Day 8

I. Last time

II All of Trig in $e^{i\theta}$

III The Disk of Convergence

I • Introduced

$$i = \sqrt{-1}$$

$$z = x + iy \quad \text{and} \quad \bar{z} = x - iy$$

• Defined multiplication:

Let $z = x+iy$ $\quad w = u + iv$

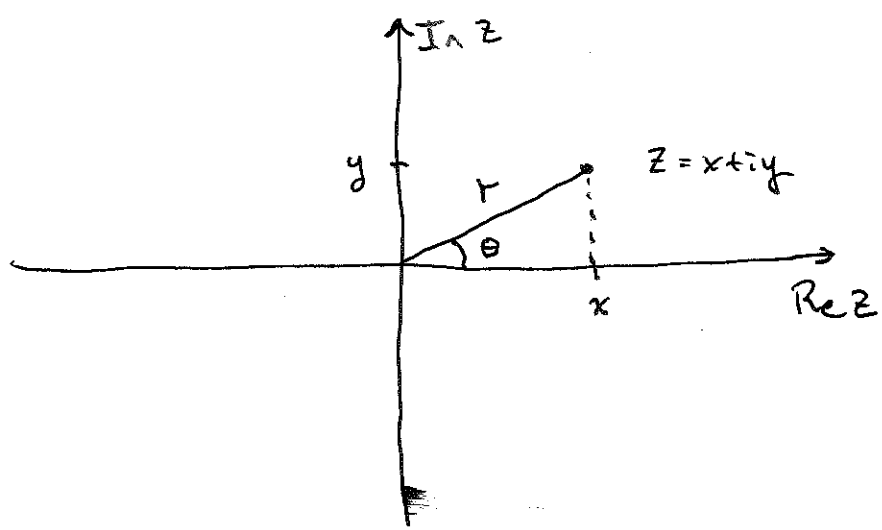
$$zw = xu - yv + i(yu + xv)$$

Division:

$$\frac{z}{w} = \frac{z\bar{w}}{w\bar{w}} = \frac{z\bar{w}}{|w|^2} = \frac{xu + yv + i(yu - xv)}{u^2 + v^2}.$$

• Graphical representation

$z = x+iy \quad |z| = \sqrt{z\bar{z}} \equiv r = \sqrt{x^2+y^2}$



II We can also introduce the angle θ into the diagram at left. Then we see

$$x = r\cos\theta$$
$$y = r\sin\theta$$

and

$$z = r(\cos\theta + i\sin\theta).$$

This representation is closely related to one more way of organizing calculations – one of the most beautiful in all of physics and mathematics.

To capture this new perspective, we need to know what we mean by

$$e^{z}$$

with z a complex number, i.e. $z \in \mathbb{C}$. (Symbol for complex #s.)

Here we get to use our idea of defining a function through its power series, we let

$$e^{z} \equiv 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \frac{1}{4!}z^4 + \cdots$$

Because we know what $z \cdot z$ means,

But these two series we know:

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

Euler's Formula

This formula is as good as it gets. Unexpected and simple, it still manages to capture all of trigonometry in one line. Master this formula, you won't regret it.

We can extend our chain of identities

the right hand side is perfectly well-defined.

Let's try this definition out on the complex number $i\theta$ to see what is new about it:

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots$$

collecting real and imag. parts

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \cdots\right)$$

now

$$\underset{\text{complex form}}{z} = \underbrace{x+iy}_{\text{cartesian form}} = \underbrace{r(\cos\theta + i\sin\theta) = re^{i\theta}}_{\text{polar forms}}$$

Ex.s: Sum of two angles identity

$$e^{i(\theta+\phi)} = \cos(\theta+\phi) + i\sin(\theta+\phi)$$

$$= e^{i\theta}e^{i\phi} = [c\theta + is\theta][c\phi + is\phi]$$

$$= c\theta c\phi - s\theta s\phi + i(c\theta s\phi + s\theta c\phi)$$

So,

$$c(\theta+\phi) + i\,s(\theta+\phi) = c\theta c\phi - s\theta s\phi + i(c\theta s\phi + s\theta c\phi)$$

We can take the real part of both sides to find

$$c(\theta+\phi) = c\theta c\phi - s\theta s\phi$$

and the imaginary part gives

$$s(\theta+\phi) = c\theta s\phi + s\theta c\phi$$

Two trig. identities out of a short calculation!

Let's try another: $(e^{i\theta})^2$

$$(e^{i\theta})^2 = (c\theta + i s\theta)^2 = c^2\theta - s^2\theta + i\,2c\theta s\theta$$

Note that taking Re and Im parts of both sides of an equation always works, so if $z = x+iy$ and $w = u+iv$ and $z = w$ then immediately

$$x = u \quad \text{and} \quad y = v$$

both follow. This is another reason complex numbers are powerful — they encode two real equations into one complex equation.

So, $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$ is a good definition, but ...

But, also

$$(e^{i\theta})^2 = e^{i2\theta} = c(2\theta) + i s(2\theta)$$

So,

$$c(2\theta) = c^2\theta - s^2\theta = 1 - 2s^2\theta$$
$$= 2c^2\theta - 1$$

and

$$s(2\theta) = 2c\theta s\theta$$

Also, taking $\theta \to \frac{1}{2}\theta$

$$c\theta = 2c^2\tfrac{\theta}{2} - 1 \Rightarrow c\tfrac{\theta}{2} = \sqrt{\frac{1+c\theta}{2}}.$$

III ... How do we define and discuss convergence for a complex infinite series?

Let

$$S_n = X_n + iY_n$$

be the partial sum of n terms of a complex series. We say that a complex series converges if

$$S = \lim_{n\to\infty} S_n = X + iY$$

is a finite complex number. Note that this amounts to

$$X = \lim_{n\to\infty} X_n \quad \text{and} \quad Y = \lim_{n\to\infty} Y_n$$

You will prove on the homework that it is still the case that absolute convergence implies convergence. Since $|z|$ is real, we can apply all our old tests to absolute convergence.

Let's consider

$$\sum_{n=1}^{\infty} n^2 (3iz)^n$$

and apply the ratio test:

$$\rho_n = \left| \frac{(n+1)^2 (3iz)^{n+1}}{n^2 (3iz)^n} \right| = \left| \frac{(n+1)^2}{n^2} 3iz \right|$$

and

$$\rho = \lim_{n\to\infty} \rho_n = |3iz|,$$

but $|3iz| = \sqrt{(3iz)(-3i\bar{z})} = \sqrt{(3z)(3\bar{z})}$

$$= |3z|.$$

So,

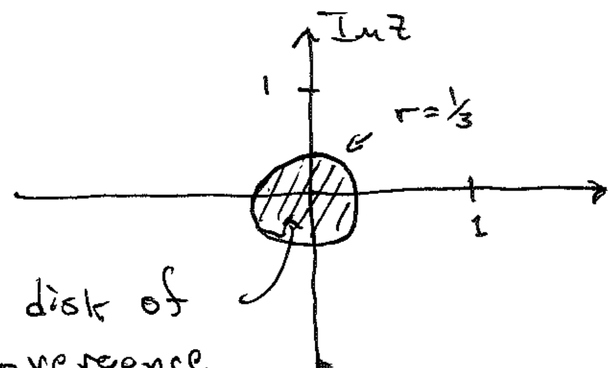
$$\rho < 1 \Rightarrow |3z| < 1$$

$$\Rightarrow |z| < 1/3.$$

Now, note that $|z| = r$, so this series converges for any complex number that has a magnitude

$$r < 1/3,$$

which is a disk of radius $1/3$



The disk of convergence for $\sum_{n=1}^{\infty} n^2 (3iz)^n$.

This is a general feature, instead of an interval of convergence complex series converge everywhere inside a disk.

Another example:

$$\sum_{n=1}^{\infty} 2^n (z+i-3)^{2n}$$

Here

$$\rho_n = \left| \frac{2^{n+1}(z+i-3)^{2n+2}}{2^n (z+i-3)^{2n}} \right| = \left| 2(z+i-3)^2 \right|,$$

which is independent of a and so,

$$\rho = |2(z+i-3)^2| < 1$$

gives the radius of convergence. We can simplify this a bit

$$|2(z+i-3)^2| = \sqrt{2(z+i-3)^2 \cdot 2\overline{(z+i-3)}^2}$$

$$= 2|z+i-3|^2$$

Then the disk of convergence can also be written as

$$|z+i-3| < \frac{1}{\sqrt{2}}$$

Graphically this is