Math Methods
Day 9

Sept. 16, 2016 Pl/3

I Last time

II Going Backwards: Inverting complex functions

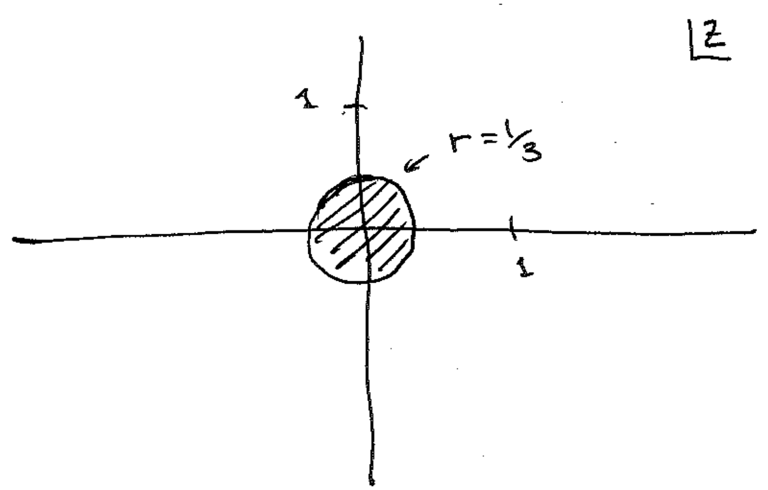
I
- Discovered Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

which completed our quartet of identities

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

- We introduced convergence of complex series and studied their disk of convergence, e.g. for $\sum_{n=1}^{\infty} n^2 (3iz)^n$ we found



Another important example is the first complex function we defined

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots$$

We have

$$\rho_n = \left| \frac{z^{n+1}}{(n+1)!} \cdot \frac{n!}{z^n} \right| = \left| \frac{z}{(n+1)} \right|$$

and so

$$\rho = \lim_{n\to\infty} \rho_n = 0 < 1 \text{ always}$$

So this series converges for all z — it has an infinite radius of convergence.

II We can solve Euler's identity for $\cos\theta$:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\overline{(e^{i\theta})} = e^{-i\theta} = \cos\theta - i\sin\theta = \overline{\cos\theta + i\sin\theta}$$

$$\Rightarrow \boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}}$$

Similarly

$$\boxed{\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$$

The complex logarithms get a bit more interesting. We still want

$$\ln z_1 z_2 = \ln z_1 + \ln z_2.$$

So,

$$\ln z = \ln(re^{i\theta}) = \text{Ln}\, r + \ln e^{i\theta}$$

$$= \underbrace{\text{Ln}\, r}_{\text{real logarithm}} + i\theta$$

This is not unique! Because θ is only defined up to

$$\theta = \theta + 2\pi n, \quad n \in \mathbb{N}.$$

This suggests that we define

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2}$$

So, e.g.,

$$\cos(i) = \frac{e^{-1} + e^{1}}{2} = \cosh(1).$$

This also suggests

$$\sinh z = \frac{e^{z} - e^{-z}}{2}$$

$$\cosh z = \frac{e^{z} + e^{-z}}{2}.$$

We call the principle value of $\ln z$ the one with $0 \le \theta < 2\pi$.

Ex:

principle value.

$$\ln(-1) = \text{Ln}(1) + i\pi$$

or $\theta = i\pi \pm 2\pi n i$ in general.

It turns out that the complex logarithm is a powerful inverse function — many other inverse functions can be written in terms of it

Recall that by inverse function we mean it undoes some function's action:

$$z = e^{w} \Rightarrow \ln z = w$$

or

$$w = \cos z \Rightarrow z = \arccos w.$$

Let's explore $z = \arccos w$ a little.
Note that we can write this as

$$w = \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

Let $u = e^{iz}$ then $u^{-1} = e^{-iz}$
and

$$\frac{u + u^{-1}}{2} = w$$

or

$$u^2 + 1 = 2wu$$

$$\Rightarrow u^2 - 2wu + 1 = 0$$

We can solve this quadratic equation

$$u = \frac{2w \pm \sqrt{4w^2 - 4}}{2} = w \pm \sqrt{w^2 - 1}$$

Recalling $u = e^{iz}$ we get,

$$e^{iz} = w \pm \sqrt{w^2 - 1}$$

$$\Rightarrow iz = \ln\left(w \pm \sqrt{w^2 - 1}\right)$$

$$\Rightarrow \boxed{\begin{array}{l} z = -i \ln\left(w \pm \sqrt{w^2 - 1}\right) \\ \hline = \arccos w \end{array}}$$

This is a very cool and original way to think about the inverse trig functions.

Ex:

$$\arccos(i) = -i \ln\left(i \pm \sqrt{i^2 - 1}\right)$$

$$= -i \ln\left(i + i\sqrt{2}\right) \quad \text{choose + sign}$$

$$= -i\left(\mathrm{Ln}(1+\sqrt{2}) + i\frac{\pi}{2}\right) \quad \text{Principal value}$$

$$= \frac{\pi}{2} - i\,\mathrm{Ln}(1+\sqrt{2})$$

$$= \frac{\pi}{2} - i(0.88137)$$

$$= \boxed{1.5708 - i(0.88137)}$$