Quiz 1

1. Derive the Taylor series for $\ln(1+x)$ about the point $x = 0$.

   In general a Taylor series takes the form
$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \cdots = \sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)}(a)(x-a)^n.$$
   For our example, $f(x) = \ln(1+x)$ we have
$$f(0) = \ln(1) = 0, \qquad f'(0) = \frac{1}{1+x}\Big|_{x=0} = 1, \qquad f''(0) = -\frac{1}{(1+x)^2}\Big|_{x=0} = -1,$$
   and each further derivative pulls down the next integer with the opposite sign, that is, $f^{(n)}(0) = (-1)^{n+1}(n-1)!$. Thus, we have
$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}(n-1)!\; x^n = \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\; x^n.}$$

2. Use the ratio test to find the interval of convergence for this series.

   For the ratio test we consider
$$\rho_n = \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{n+1}\frac{n}{x^n}\right| = \left|\frac{nx}{n+1}\right|.$$
   Taking the limit then gives
$$\rho = \lim_{n\to\infty} \rho_n = \lim_{n\to\infty} \left|\frac{nx}{n+1}\right| = |x|.$$
   Then according to the ratio test, the series certainly converges in the interval $|x| < 1$, but what of the boundaries of this region?

3. Bonus: Determine whether the series converges at both boundaries of the interval of convergence.

   To figure these out, we plug in the boundary cases $x = \pm 1$. For $x = -1$,
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\;(-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n} = -\sum_{n=1}^{\infty} \frac{1}{n},$$
   but this is negative the harmonic series and we know that series diverges. For $x = 1$, the series is
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n},$$

which is an alternating series. We can run the alternating series test on it. Certainly $|a_{n+1}| < |a_n|$ and also

$$\lim_{n\to\infty} \frac{(-1)^{n+1}}{n} = 0,$$

so this alternating series converges. Hence the full interval of convergence for this series is $\boxed{-1 < x \leq 1.}$