Homework 2

Due Sunday, February 10th at 5pm

Read Boas Ch. 6, sections 1 to 6, and all of Ch. 5.

1. Much as we defined bivectors through the wedge product, we can also define trivectors (actually at this point most people stop naming things this way and just call them k-vectors, or in this case 3-vectors). Concretely, given three vectors $\vec{a}$, $\vec{b}$, and $\vec{c} \in \mathbb{R}^3$ the wedge product of all three is
$$\vec{a} \wedge \vec{b} \wedge \vec{c};$$
and as before it is antisymmetric in each switch of the vectors
$$\vec{a} \wedge \vec{b} \wedge \vec{c} = -\vec{b} \wedge \vec{a} \wedge \vec{c} = -\vec{a} \wedge \vec{c} \wedge \vec{b} = \vec{c} \wedge \vec{a} \wedge \vec{b}, \quad \text{etc.},$$
and is linear in each of the three slots. We call the space of 3-vectors $\bigwedge^3 \mathbb{R}^3$. (**a**) Find a basis for this space?

   (**b**) What is the dimension of this space?

   (**c**) Write three general vectors $\vec{a}$, $\vec{b}$, and $\vec{c} \in \mathbb{R}^3$ in the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and compute
$$\vec{a} \cdot (\vec{b} \times \vec{c}).$$
Also compute $\vec{a} \wedge \vec{b} \wedge \vec{c}$ in the basis for 3-vectors you found in part (**a**). Compare the results.

2. (**a**) Write out a basis for bivectors in $\bigwedge^2 \mathbb{R}^4$.

   (**b**) In $\bigwedge^2 \mathbb{R}^4$ something new can happen that wasn't possible in $\bigwedge^2 \mathbb{R}^3$. Namely, there exist perfectly good bivectors that cannot be written as the wedge product of two vectors in $\mathbb{R}^4$. To illustrate this prove that the bivector
$$B = \mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_3 \wedge \mathbf{e}_4,$$
cannot be written as the product of two vectors in $\mathbb{R}^4$.

3. (**a**) Find the dimension of $\bigwedge^2 \mathbb{R}^n$, where n is any integer greater than or equal to 2. If you don't have an idea of how to do this at first, build it up through examples. Find the dimensions of $\bigwedge^2 \mathbb{R}^4$, $\bigwedge^2 \mathbb{R}^5$, and $\bigwedge^2 \mathbb{R}^6$. Can you spot the pattern now?

   (**b**) Find the dimension of $\bigwedge^3 \mathbb{R}^n$ for $n \geq 3$.

   (**c**) Find the dimension of $\bigwedge^k \mathbb{R}^n$ for $k \leq n$. [Suggestion: If you have not met the choose function $\binom{a}{b}$, you can answer part (c) with the phrase "I haven't used the choose function before" and that will count as a full solution for (c) and (d).]

   (**d**) Fix $n = 6$ and consider $k = 1, \ldots, 6$. Write out the dimension of $\bigwedge^k \mathbb{R}^6$ for each of these k. [When $k = 1$, $\bigwedge^1 \mathbb{R}^6$ just means the vectors space $\mathbb{R}^6$.] Do you notice something interesting about these dimensions?

4. In this problem you will complete the proof of the $BAC - CAB$ identity that we began in class. (Of course, since you are trying to prove the identity, you cannot use it in this problem.) In class we argued that

$$\vec{A} \times (\vec{B} \times \vec{C}) = c_1 \vec{B} + c_2 \vec{C} \tag{1}$$

where c_1 and c_2 are scalar coefficients that might depend on $\vec{A}$, $\vec{B}$, and $\vec{C}$.

(**a**) Use properties of the cross product to argue that $\vec{A}$ is orthogonal to $\vec{A} \times (\vec{B} \times \vec{C})$.

(**b**) Use your result from (**a**) and our result from class, Eq. (1), to prove that

$$\frac{c_1}{c_2} = -\frac{\vec{A} \cdot \vec{C}}{\vec{A} \cdot \vec{B}}.$$

This alone is not enough to conclude that $c_1 = \vec{A} \cdot \vec{C}$ and $c_2 = -\vec{A} \cdot \vec{B}$, however, there is a nice argument you can make. The trouble is that we have two unknown coefficients and only one equation relating them. Nonetheless we can say that

$$c_1 = \lambda \vec{A} \cdot \vec{C} \qquad \text{and} \qquad c_2 = -\lambda \vec{A} \cdot \vec{B},$$

where λ is a real constant, because in this form c_1/c_2 still gives the right ratio.

(**c**) Then we have

$$\vec{A} \times (\vec{B} \times \vec{C}) = \lambda[\vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})]$$

and only need to fix λ. To be an identity this equation must hold for all choices of vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$. Evaluate both sides of the equation for the particular choice $\vec{A} = \hat{x}$, $\vec{B} = \hat{x}$, and $\vec{C} = \hat{y}$ and fix the value of λ.

This gives a complete proof of the $BAC - CAB$ rule.

5. Boas 6.3.12. [Hint: If this is a long calculation, you probably want to try a different technique.]

6. Boas 6.3.13. If you interpret the vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ as the edge vectors of our tetrahedron, as in class, what is this problem telling us?

7. (**a**) Boas 6.3.16 and (**b**) Boas 6.3.17.