Homework 4

Due Sunday, March 3rd at 6pm

Read Boas Ch. 7, §§1-10.

1. (**a**) A nice warm up for doing two-dimensional integrals in the polar plane is to find the area of a circle of radius R. Recall that we proved in class that the area of a region σ, described in polar coordinates, is given by
$$A(\sigma) = \iint_\sigma r d\theta dr.$$
Identify the θ- and r-limits of integration and carry out this integral to prove that the area of a circle is $A_\circ = \pi R^2$.
Now, repeat the process for a more challenging example, the ellipse of semi-major axis a and semi-minor axis b.
(**b**) First you will need to find the polar form for the equation of the ellipse. You can do this by starting with the equation
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$
and substituting $x = r\cos\theta$ and $y = r\sin\theta$ and then solving for $r(\theta)$. The curve $r(\theta)$ provides one of the boundaries of integration for your integral.
(**c**) Use the formula you derived in part (**b**) to find the area of an ellipse. Here you will have to not only think about the bounds of integration, but also the appropriate order of integration.

2. Boas (**a**) 5.2.27 & (**b**) 5.2.28

3. Boas (**a**) 5.2.29 & (**b**) 5.2.31

4. In the last problem of homework 3 you derived a new coordinate system for the plane. These are called parabolic coordinates. Find the Jacobian that connects the area element of Cartesian coordinates to that of parabolic coordinates
$$dxdy = |J| d\sigma d\tau,$$
using the Jacobian determinant method we derived in class.

5. Consider cylindrical coordinates for three-dimensional space. These coordinates are the same as polar coordinates in the xy-plane and the third coordinate is the z-coordinate along the z-axis; in sum they are (r, θ, z). (**a**) Find the Jacobian connecting the volume element of $\mathbb{R}^3$ in Cartesian coordinates to that in cylindrical coordinates
$$dV = dxdydz = |J| dr d\theta dz,$$
using the Jacobian determinant method. (**b**) Derive this connection again using the geometrical method. [Please don't refer to the book to solve this problem. Do it on your own.]

6. As a reminder, if A and B are 2×2 matrices, then the product AB is defined by

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}.$$

   (**a**) Prove that $\det(AB) = \det A \det B$ for 2×2 matrices. This is also true for $n \times n$ matrices, but you don't need to prove it. If you have proved this result before, it is ok with me if you write "I have proved this before." instead of doing it again.
   (**b**) Boas 5.4.18

7. Boas (**a**) 5.4.14, (**b**) 5.4.19 [Feel free to use the result of Problem 6. here.], and (**c**) 5.4.20