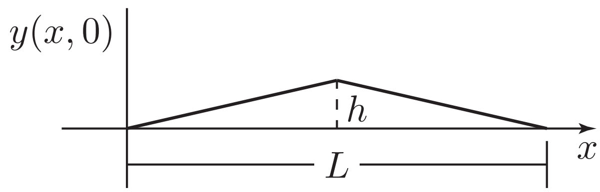
Homework 6

Due Sunday, March 17th at 6pm

Read Boas Ch. 6, §§7-11.

1. Consider a plucked violin string, as discussed in class we can find $y(x,t)$, the displacement at time t of any point x of the vibrating string from its equilibrium position.

   (**a**) Recall that to solve this problem, you first need to expand the function $y(x,0)$, whose graph is the initial shape of the string, in a Fourier sine series. Find this series if a string of length L is pulled aside a small distance h at its center, as shown.

   (**b**) Use the form of the general solution that appeared in our class notes, to find the general solution to the wave equation, $y(x,t)$, for this particular initial condition.

2. Use exponential Fourier series to find expansions for problem 7.5.9 and 7.5.11 and verify in each case that the answer is equivalent to your previous results.

3. Boas 7.7.12

4. Recall that we define a complex number z in terms of its real, x, and imaginary, y, parts through

   $$z = x + iy,$$

   where $i = \sqrt{-1}$. The complex conjugate $\bar{z}$ is defined as $\bar{z} = x - iy$. More generally, you can prove that the complex conjugate of any complex function is simply the same function except with all the i's replaced by $-i$'s. The real part of z is just $\operatorname{Re} z = \frac{1}{2}(z + \bar{z})$ and the imaginary part is $\operatorname{Im} z = \frac{1}{2i}(z - \bar{z})$.

   (**a**) Show that $y(x,t) = Ae^{i(kx-\omega t)}$ is a solution to the classical wave equation if and only if $v = \omega/k$. You can take the constant A to be real.

   (**b**) Take the real part of the function from part (**a**), is the resulting wave moving to the right or to the left? Now, take the imaginary part, is the resulting wave moving to the left or to the right? What is the difference between these two waves? Can you make this difference completely quantitative?

5. In class we defined the multivariable operations of gradient and divergence

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

and

$$\vec{\nabla}\cdot\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}.$$

A third type of vector derivative you can define is the *curl*, which is defined by

$$\vec{\nabla}\times\vec{V} \equiv \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\hat{x} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\hat{y} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{z}.$$

Note that you have to interpret the first equality carefully and think of the partial derivative operators as acting on the elements of the last row.

(**a**) In class and on the homework we have studied the three vector fields $\hat{r}$, $\hat{\theta}$ and $\hat{z}$. These vector fields describe the basis vectors for cylindrical coordinates. Express all three of these vectors fields in Cartesian components $\hat{x}, \hat{y}$, and $\hat{z}$ and in Cartesian coordinates x, y, and z (yes, one of them will be trivial). Sketch each of the vector fields in 3D.

(**b**) Calculate the divergence of each of these vector fields. Do your results make sense given the sketches you made above.

(**c**) Calculate the curl of each of these vector fields. Make a guess about what the curl tells you about a vector field. It is fine that this is just a guess, we'll talk about it more in a few weeks.

6. (**a**) Use the BAC-CAB rule to derive an identity for how to simplify the vector derivatives in the expression

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{V}).$$

Your answer should be expressed in terms of the divergence, the gradient and a new vector derivative

$$\vec{\nabla}\cdot\vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

which we call the Laplacian and often abbreviate with the symbol $\vec{\nabla}\cdot\vec{\nabla} = \nabla^2$.

(**b**) You may have studied electricity and magnetism before and encountered Maxwell's equations. (No problem if you haven't, this exercise isn't really about electromagnetism, it's about what we've been doing in class.) Maxwell's equations in the vacuum (when the charge density ρ and current $\vec{J}$ vanish, $\rho = \vec{J} = 0$) can be written as

$$(i)\quad \vec{\nabla}\cdot\vec{E} = 0, \qquad (ii)\quad \vec{\nabla}\cdot\vec{B} = 0,$$

$$(iii)\quad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \qquad (iv)\quad \vec{\nabla}\times\vec{B} = \mu_o\epsilon_o\frac{\partial\vec{E}}{\partial t},$$

where $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field and μ_o and ϵ_o are the permeability and permittivity of free space respectively.

By taking the partial time derivative of Eq. (iii), using your result from part (**a**), and combining these equations prove that the magnetic field satisfies the wave equation. Repeat this derivation with the right changes to show that the electric field satisfies the wave equation.

(**c**) By comparing the wave equation you derived in part (**b**) to the standard form of the classical wave equation, find the velocity of electromagnetic waves. By looking up the values of the constants that appear in your expression calculate a value in meters per second for this speed. Is your result surprising?