Math Methods II

Day 1

Today

0. Review Syllabus

I. Doing 2D & 3D Geometry with Vectors & the Vector Space axioms

I. Table of Vector Space Axioms

A set of objects V is a vector space if its elements $\vec{u}, \vec{v}, \vec{w}$ etc satisfy:

1. Closure: Given $\vec{u}, \vec{v} \in V$ then $\vec{u}+\vec{v} \in V$

2. Vec addition + is (a) commutative: $\vec{u}+\vec{v} = \vec{v}+\vec{u}$
(b) associative: $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$

3. (a) $\exists$ a vec $\vec{0}$ s.t. $\vec{0}+\vec{v} = \vec{v}+0 = \vec{v}$, $\forall \vec{v} \in V$.
(b) For every $\vec{v} \in V$ $\exists$ $(-\vec{v}) \in V$ s.t. $\vec{v}+(-\vec{v}) = \vec{0}$.

There is a notion of scalar multiplication of vectors such that:

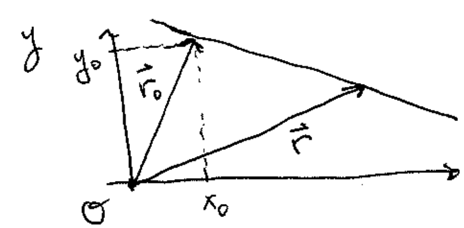
4. (a) $k(\vec{u}+\vec{v}) = k\vec{u}+k\vec{v}$
(b) $(k_1+k_2)(\vec{v}) = k_1\vec{v}+k_2\vec{v}$
(c) $(k_1 k_2)\vec{v} = k_1(k_2\vec{v})$
(d) $0 \cdot \vec{v} = \vec{0}$, and $1 \cdot \vec{v} = \vec{v}$.

<u>Additional Structure:</u> Basis $\{\hat{i}, \hat{j}, \hat{k}\}$ or $\{\hat{x}, \hat{y}, \hat{z}\}$.

Inner product

$\vec{A} \cdot \vec{B} = \sum_i A_i B_i$

Let's begin with a physical example. I love billiards, so, let's say a pool ball is flying along the straight line pictured below



Take your origin of coordinates at the lower left-hand pocket. How do we describe the motion of the pool ball using vector equations? Say we know that initially it is at position

$$\vec{r}_0 = (x_0, y_0) = x_0\hat{x} + y_0\hat{y} = x_0\hat{i} + y_0\hat{j}$$

and the pool cue struck it in the direction $\vec{A} = (a_1, a_2)$, then what is $\vec{r}(t)$?

Well, $\vec{r} - \vec{r}_0 \propto \vec{A}$, let's call the proportionality factor t. Then

$$\vec{r} - \vec{r}_0 = t\vec{A} \quad \text{or} \quad \vec{r} = \vec{r}_0 + t\vec{A}.$$

We can also write this in the more symmetrical form

$$\frac{x - x_0}{a_1} = t = \frac{y - y_0}{a_2}$$

Notice that this would have worked in any number of dimensions.

What is the velocity of the ball?

$$v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}$$

$$\vec{v} = (v_x, v_y) = \frac{d}{dt}(x, y) = \frac{d}{dt}\vec{r}$$

How many eqn.s is this? What are they?

$$\left.\begin{aligned} x &= x_0 + a_1 t \\ y &= y_0 + a_2 t \end{aligned}\right\} \text{2 eqn.s}$$

This is the "parametric" form of a line; here t is the parameter. Can we recover the standard $y = mx + b$ form?

$$t = \frac{x - x_0}{a_1}$$

$$\Rightarrow \quad y = \frac{a_2}{a_1}(x - x_0) + y_0 = \overbrace{\frac{a_2}{a_1}}^{m} x + \overbrace{y_0 - \frac{a_2}{a_1}x_0}^{b}$$

so,

$$\vec{v} = \frac{d}{dt}(\vec{r}_0 + t\vec{A}) = \vec{A}$$

This makes sense since that was the direction the ball was moving.

What is the angular momentum of the ball with respect to (w.r.t.) O?

$$\vec{l} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$$