Math Methods

Day 10

Today

I Last time

II An example of a multiple integral

III A first look at longitudinally coupled oscillators

I We set up and considered multiple integrals. My suggestions were to:

- invest time in setting up the integral
- Carefully consider and sketch the region of integration.
- check whether exchanging the order of integration will simplify a calculation

We also investigated Jacobians.

In a coordinate transformation from (x,y) to (s,t) coordinates the Jacobian changes the integration measure by

$$dx\,dy = |J|\,ds\,dt$$

where

$$J = \det \frac{\partial(x,y)}{\partial(s,t)} = \det \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix}$$

We proved this result using differentials and the wedge product. We also found the Jacobian factor for Spherical coords $(x,y,z) \to (r,\theta,\phi)$:

$$J = r^2 \sin\theta$$

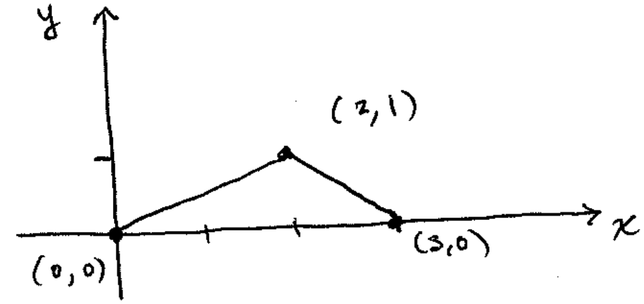
II Answers to many problems in the book are at the back.

Let's do a multiple integral.

We'll do Boas 5.2.13:

$\iint 2xy\,dx\,dy$ over the triangle with vertices $(0,0)$, $(2,1)$, and $(3,0)$.

So, first let's sketch the region of integration:



the y bounds of integration change at $(x,y) = (2,0)$. We also see that we should do the y-integrals first since their bounds of integration depend on the x-position. In the first region we have

$$y = mx + b$$

or

$$y = \left(\frac{1-0}{2-0}\right)x + 0 = \frac{1}{2}x$$

In this problem it makes sense to break the integral into two pieces: the first piece is over  and the second is over . This is because for the upper bound. In the second region it is

$$y = \frac{(0-1)}{(3-2)}x + b$$

and

$$\underset{\text{insert}}{1} = -1(\underset{(2,1)}{2}) + b \Rightarrow b = 3,$$

so, finally

$$y = -x + 3.$$

Using these bounds our integral

becomes

$$\iint 2xy\,dx\,dy =$$

$$\int_{x=0}^{x=2}\int_{y=0}^{y=\frac{1}{2}x} 2xy\,dy\,dx + \int_{x=2}^{x=3}\int_{y=0}^{y=-x+3} 2xy\,dy\,dx$$

$$= \int_{x=0}^{2} 2x\left(\frac{y^2}{2}\right)_0^{\frac{1}{2}x} dx + \int_{x=2}^{3} 2x\left(\frac{y^2}{2}\right)_0^{-x+3} dx$$

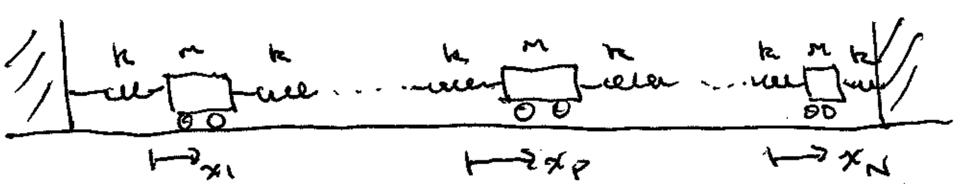
$$= \int_{x=0}^{2} x\,\frac{x^2}{4}\,dx + \int_{x=2}^{3} x\,(-x+3)^2\,dx$$

Then, the total integral is

$$\iint 2xy\,dx\,dy = 1 + \frac{3}{4} = \boxed{\frac{7}{4}}$$

$\triangle$

III Moving forward we are going to build up wave physics using a physical model of oscillating masses



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The 1st of these integrals is

$$\int_0^2 \frac{1}{4}x^3\,dx = \frac{x^4}{16}\Big|_0^2 = \frac{16}{16} - 0 = 1$$

and the 2nd is

$$\int_2^3 (x^3 - 6x^2 + 9x)\,dx$$

$$= \left[\frac{x^4}{4} - 2x^3 + \frac{9}{2}x^2\right]_2^3$$

$$= \frac{81}{4} - 54 + \frac{81}{2} - 4 + 16 - 18 = \frac{3}{4}$$

Consider N carts of equal mass connected by springs of strength k. The equation of motion of the p^{th} cart is

$$\frac{d^2x_p}{dt^2} = \overbrace{-k(x_p - x_{p-1})}^{\text{Spring to left of cart is amount stretched}} \overbrace{-k(x_p - x_{p+1})}^{\text{Spring to right is amount stretched}}$$

$$= -k(2x_p - x_{p-1} - x_{p+1})$$

In the coming days we will study the solutions to this eqn. where all carts oscillate at the <u>same</u> frequency.