Math Methods
Day 11

<u>Today</u>

I Last time

II Summary of N longitudinally coupled oscillators

III A little exam review

I • We proved that the change of coord.s $x = x(s,t)$ $y = y(s,t)$ is accompanied by a change in measure

$$dx\,dy = |J|\,ds\,dt$$

with

$$J = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} \equiv \frac{\partial(x,y)}{\partial(s,t)}$$

• For polar coord.s

$$dx\,dy = r\,dr\,d\theta$$

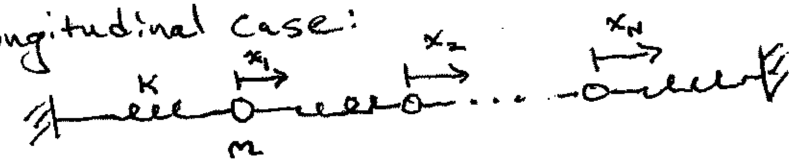
and for spherical coords

$$dx\,dy\,dz = r^2 \sin\theta\,dr\,d\theta\,d\phi.$$

• Showed that one use for the wedge product was apparent when we derived the expression for J.

II <u>N Coupled Oscillators</u>

Longitudinal case:



The eqns. of motion (EoM) are:

$$m\frac{d^2x_p}{dt^2} = -K\left(2x_p - x_{p-1} - x_{p+1}\right) \begin{cases} p = 1, 2, \dots, N \\ x_0 = 0,\ x_{N+1} = 0 \end{cases}$$

These equations had a special set of solutions where all the masses oscillated at the <u>same</u> frequency:

These are the

<u>Normal Mode Solutions</u>:

$$x_p(t) = A \sin\left(\frac{p n \pi}{N+1}\right) \cos(\omega_n t)$$

where

N = # of masses

p = the particular mass

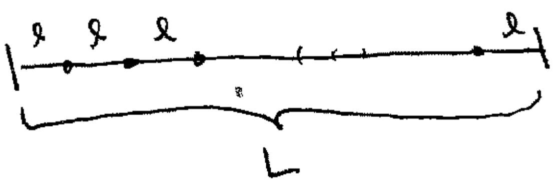
n = the mode with freq. ω_n

and

$$\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right) ; \quad \omega_0 = \sqrt{\frac{k}{m}}$$

(Here we drop a phase constant, but that's fine because they are in phase unless maybe they are off by 180° and this would just switch the sign of A, the Amplitude.)

The spacing btwn neighboring masses (AT EQUILIBRIUM) is ℓ



So, $\ell = \frac{L}{N+1}$. Let x be the equilibrium position of the pth mass (from the left end), then

$$x = p\ell = p\frac{L}{N+1} \quad \text{or} \quad \frac{p}{N+1} = \frac{x}{L}$$

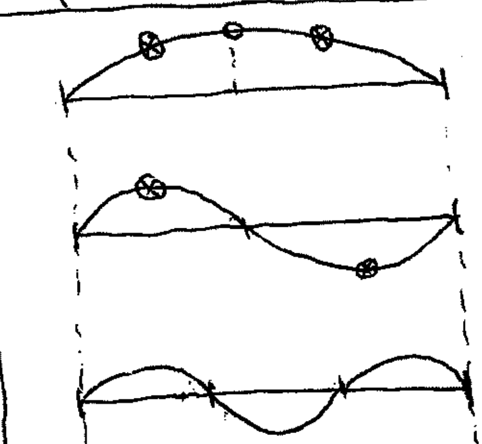
<u>Visualizing the Motion & Continuum</u>

The "Amplitude" of the p^{th} mass:

$$A_p = A \sin\left(\frac{p n \pi}{N+1}\right)$$

and

$$A_p = A \sin\left(n\pi \frac{x}{L}\right)$$

A_p	n	A_p vs. x [$N=1$ ○ ; $N=2$ ⊗]
$A \sin\left(\frac{\pi x}{L}\right)$	1	
$A \sin\left(\frac{2\pi x}{L}\right)$	2	
$A \sin\left(\frac{3\pi x}{L}\right)$	3	

III We haven't done many examples of computing divergences. To compute a divergence we need a vector field, let's imagine

$$\vec{v} = xy\,\hat{x} + yz\,\hat{y} + zx\,\hat{z}$$

$$= (xy,\ yz,\ zx)$$

And, let's also imagine we want to evaluate the divergence at the point $\vec{r} = (1,2,3) = 1\hat{x} + 2\hat{y} + 3\hat{z}$

We denote this

$$\vec{\nabla}\cdot\vec{v}\Big|_{\vec{r}} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(xy, yz, zx)$$

means evaluated at

$$= \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(zx)$$

$$= y + z + x\Big|_{\vec{r}}$$

$$= 2 + 3 + 1 = \boxed{6.}$$

This means that there is a net flow of $\vec{v}$ out of the point $\vec{r} = (1,2,3)$ and the net flow has a magnitude of 6.