Math Metods
Day 12

Today
I. Last Time
II. Transverse Coupled Oscillators
III. Continuum Limit
IV. Waves & The Wave Equation

I. Last Time

- N coupled longitudinal oscillators have normal mode solutions

$$x_p(t) = A \sin\left(\frac{pn\pi}{N+1}\right)\cos(\omega_n t),$$

  where p indexes the oscillators $(p = 1, \ldots, N)$ and n indexes which normal mode is being considered and $(n = 1, \ldots, N)$. The normal mode frequencies are

$$\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right); \qquad \omega_0 = \sqrt{\frac{k}{m}}.$$

- We reviewed the direct calculation of a divergence $\boldsymbol{\nabla} \cdot \mathbf{v}$.

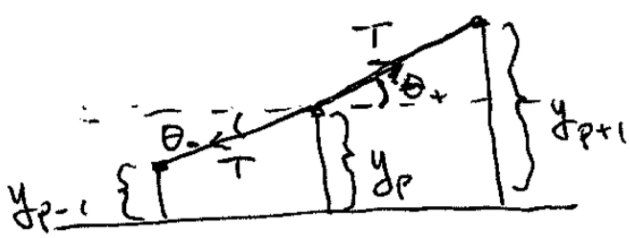
II. Transverse Coupled Oscillators

Consider a massless string under tension T, with N masses at regular intervals.

Figure 1: A collection of N masses, each of mass m, at equilibrium under a tension T.

Figure 2: We take the displacement of the p^{th} mass to be completely vertical and described by $y_p(t)$.

Let $y_p(t)$ be the (transverse) displacement of the p^{th} mass. Once again $\ell = \frac{L}{N+1}$ and for small displacements $(y_p << \ell)$ we have:



The force on the p^{th} mass, in the transverse direction, is

$$F = T\sin\theta_+ - T\sin\theta_-.$$

For small angles θ,

$$\sin\theta \approx \theta \approx \tan\theta$$

and so,

$$\begin{aligned} m\frac{d^2 y_p}{dt^2} &\approx T(\tan\theta_+ - \tan\theta_-) \\ &= T\left[\frac{y_{p+1} - y_p}{\ell} - \frac{y_p - y_{p-1}}{\ell}\right] \\ &= -\frac{T}{\ell}[2y_p - y_{p+1} - y_{p-1}]. \end{aligned}$$

This is the same as in the longitudinal case! We just need $k \to \frac{T}{\ell}$, so

$$\boxed{\omega_0 = \sqrt{\frac{T}{\ell m}}.}$$

Hence,

$$y_p(t) = A \sin\left(\frac{pn\pi}{N+1}\right)\cos(\omega_n t)$$

with

$$\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right) \qquad \text{and} \qquad \omega_0 = \sqrt{\frac{T}{\ell m}}.$$

III. Continuum Limit

Let's examine the limit as $N \to \infty$. We have

$$\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right) \overset{N\to\infty}{\approx} 2\sqrt{\frac{T}{\ell m}}\frac{n\pi}{2(N+1)}.$$

Recall $\ell = \frac{L}{N+1}$, so

$$\begin{aligned}\omega_n = \sqrt{\frac{T}{\ell m}}\frac{n\pi}{N+1} &= n\pi\sqrt{\frac{T}{mL}(N+1)}\frac{1}{N+1} \\ &= n\pi\sqrt{\frac{T}{mL(N+1)}}.\end{aligned}$$

Let $\mu = \frac{Nm}{L}$, the linear mass density. Then $Nm = \mu L$ and

$$\omega_n = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}},$$

where we have neglected the 1 in $(N+1)$ since we have taken N large. Now, let $x = p\ell$ (the position of the p^{th} mass), then

$$x = p\frac{L}{N+1} \implies \frac{p}{N+1} = \frac{x}{L}.$$

So that for $N \to \infty$ transverse oscillators

$$y_x(t) = y(x,t) = A\sin\left(n\pi\frac{x}{L}\right)\cos(\omega_n t + \phi_n),$$

where the first equality expresses our conceptual change from thinking of p as indexing which mass to thinking of x as labeling the position along the continuous string and

$$\omega_n = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}}.$$

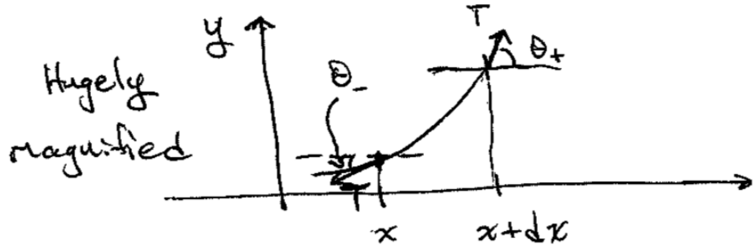
IV. Waves & The Wave Equation

Waves: Consider a string with mass per unit length μ, under tension T, of length L, and fixed at both ends. Look at transverse oscillations. We would like to find

$$y(x, t).$$

Figure 3: A taut string pinned down at both ends.

The Wave Equation: Apply Newton's 2nd law to the segment pictured below:



The net force (in the transverse direction) is

$$F = T\sin\theta_{+} - T\sin\theta_{-}.$$

So,

$$\begin{aligned} m\frac{\partial^2 y}{\partial t^2} &\approx T(\tan\theta_{+} - \tan\theta_{-}) \\ &\approx T\left(\frac{\partial y}{\partial x}\bigg|_{x+dx} - \frac{\partial y}{\partial x}\bigg|_{x}\right). \end{aligned}$$

⌈ Note: $f(x+dx) - f(x) \approx \frac{df}{dx}dx$. ⌋ So, the right hand side of our last expression simplifies to

$$T\left(\frac{\partial y}{\partial x}\bigg|_{x+dx} - \frac{\partial y}{\partial x}\bigg|_{x}\right) \approx T\frac{\partial^2 y}{\partial x^2}dx.$$

Our small segment of string has mass $m = \mu dx$, so that

$$\begin{aligned} \mu \cancel{dx}\frac{\partial^2 y}{\partial t^2} &= T\frac{\partial^2 y}{\partial x^2}\cancel{dx} \\ \implies &\boxed{\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2}.} \end{aligned}$$

This is the classical wave equation:

$$\boxed{\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2},}$$

where in our present case

$$v = \sqrt{\frac{T}{\mu}}.$$